Mathematics for Young Children (M4YC) Literature Review

Early mathematics: Challenges, possibilities, and new directions in the research

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Original publication: June 2012

Revised and updated: December 2016

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Foreword

In developing this literature summary, a comprehensive review of over 500 articles was conducted in 2011-2012. The goal was to provide a synthesis of existing literature on mathematics for young children with a focus on the fields of education and educational research.

The following criteria were considered as part of the research selection process for the literature review:

- treatment and control group studies with randomized field trials or quasi-experimental designs;
- longitudinal studies;
- a combination of quantitative and mixed methods studies with large populations, and highly descriptive studies involving smaller populations;
- research from academics who are well established in their field;
- peer-reviewed articles from top tier journals (blind peer review with high rankings).

In 2016, this literature review was updated to include the most up-to-date references and key developments in the research literature that occurred between 2012 and 2016, including the Math for Young Children research conducted in Ontario under the supervision of Principal Investigator Dr. Cathy Bruce. When the original literature review was developed, the authors were looking at early mathematics generally. In particular, research between 2012 and 2016 has been pointing to the key importance of spatial reasoning for children's mathematical development. This is the most noteworthy update to this literature review.

Updated sections include:

This literature review provides the groundwork for additional Canadian research in the field of mathematics for young children, a burgeoning area of interest for educators and researchers alike. There is much yet to learn.

Introduction

> *The human mind inevitably comprehends the world in mathematical terms (among others). Children's informal and invented mathematics contains on an implicit level many of the mathematical ideas that teachers want to promote on a formal and explicit level. These ideas may be innate, constructed for the purpose of adaptation, or picked up from an environment that is rich in mathematical structure, regardless of culture. Teachers should attempt to uncover the mathematical ideas contained in their students' thinking because much, but not all, of the mathematics curriculum is immanent in children's informal and invented knowledge. This mathematical perspective requires a focus not only on the child's constructive process but also on the mathematical content underlying the child's thinking. Teachers then can use these crude ideas as a foundation on which to construct a significant portion of classroom pedagogy. In doing thus, teachers should recognize that children's invented strategies are not an end in themselves. Instead, the ultimate goal is to facilitate children's progressive mathematization of their immanent ideas. Children need to understand mathematics in deep, formal, and conventional ways.* (Ginsburg & Seo, 1999, 113)

In recent years, a wealth of research and reports have been produced that, despite coming from different fields and using different frameworks, have all reached the same conclusion: all children should have access to high quality mathematics instructions and experiences in the early years. These conclusions come from different perspectives: psychological, developmental and educational as well as the neurosciences. This literature review will look at how these recommendations came to be, and how, in spite of emerging from a pendulous historical debate over what – and even *whether* – math instruction should be available to young children, the overwhelming conclusion from fields with an interest in the development of young children is that high quality math instruction should indeed be an important focus in the early years. This document synthesizes and extends these findings.

Section 1: Young children's informal mathematics

For at least the past 180 years, educators have debated the value or appropriateness of explicit teaching of mathematics to young children – on one side are those who believe that it is inappropriate to teach mathematics to young children, and on the other side are the proponents of mathematics education who believe that children are capable of complex mathematical thinking.

The most current research persuasively argues that the question of whether or not young children should be taught mathematics is immaterial because young children already do mathematics and think about mathematics in their day-to-day world. For Ginsburg, Lee & Boyd (2008) "[t]he question of whether young children are 'ready' to learn mathematics is beside the point... Learning mathematics is a 'natural' and developmentally appropriate activity for young children" (5).

Everyday or *informal* mathematical experiences engender mathematical thinking in even very young children as they interact with their world. Research over the past 25 years has shown that "nearly from birth to age 5, young children develop an *everyday mathematics* – including informal ideas of more and less, taking away, shape, size, location, pattern and position – that is surprisingly broad, complex, and sometimes sophisticated" (Ginsburg, Lee & Boyd, 2008, 3). Furthermore, young children have a "spontaneous and sometimes explicit interest in mathematical ideas" (Ginsburg, Lee & Boyd, 2008, 3). Ginsburg, Lee & Boyd (2008) also cite research results from the observation of children in natural environments showing that young children spontaneously count (even up to relatively large numbers) and show interest in quantities ("how many" or "how much"). Children show persistent interest in comparing heights of different towers, of exploring and creating patterns, shapes and symmetry. (Irwin & Bergham, 1992; Saxe, Guberman & Gearhart, 1987). Ginsburg, Lee & Boyd (2008) also explain how mathematics permeates children's spontaneous play.

According to Sarama & Clements (2008), "children of all ages have some knowledge of mathematics," (68) including infants as mentioned previously. Ninety-four percent of children can count to ten and recognize basic shapes by school age (West, Denton & Germino-Hausken, 2000; Clements, 1999). The majority of children starting kindergarten can count small sets of objects, solve problems involving small amounts and share small groups of objects equally between two people (Baroody & Wilkins, 1999; Hunting, 1999). Furthermore, Ginsburg, Pappas & Seo (2001) observed that pre-school children's self-selected free play involved mathematics content 50% of the time.

Contributions of cognitive and neuroscience to our understanding of early number sense development

A significant body of research in the area of early number sense merits further discussion. What is number sense, and how do children acquire it? Number sense (also referred to as number competence or number knowledge in the literature) is variously defined and operationalized through skills identified by a number of mathematics education researchers. Most seem to agree that at its most basic, number sense involves an understanding of numbers and numerical relationships (Jordan & Levine, 2009). Cognitively, this involves seeing collections of objects as “sets of individuals,” representing a group composed of individual units (Spelke, 2003 in Jordan & Levin, 2009). In *How the Brain Learns Mathematics*, Sousa (2008) traces the history of the term ‘number sense’, from Tobias Danzig (1967) who described it as a person’s ability to recognize that a quantity in a small collection has changed, to Keith Devlin (2000) who suggested that number sense was comprised of two main abilities: to compare the sizes of two collections simultaneously and to recall numbers of successive objects. Gersten, Jordan & Flojo (2005) also trace the defining characteristics of number sense in the literature, while noting that “no two researchers have defined number sense in precisely the same fashion” (296). They cite Kalchman, Moss & Case (2001) who operationalized number sense to include: “(a) fluency in estimating and judging magnitude, (b) ability to recognize unreasonable results, (c) flexibility when mentally computing, (d) ability to move among different representations and to use the most appropriate representation (297). Yet others (Okamoto, in Gersten, Jordan & Flojo, 2005) have noted two important aspects to young children’s mathematics proficiency in kindergarten by distinguishing between the ability to count and the ability to understand quantities. These components of number sense are interestingly not directly linked, as in the case of students who can accurately count to five but who are unable to say which of two or four is the bigger number (Okamoto & Case, 1996, in Gersten, Jordan & Flojo, 2005).

Geary (1996) argues that all children possess “biologically primary” abilities including number and basic geometry (see Ginsburg, Lee & Boyd, 2008, pg 4). Babies, for instance, can discriminate between two collections of different quantities (Lipton & Spelke, 2003; Starkey & Cooper, 1980; Strauss & Curtis, 1981). *Subitizing* (from the Latin for “sudden”) refers to the innate ability to visually process objects of four or less (Sousa, 2008). Counting, on the other hand, is necessary for larger collections, the quantity of which cannot be visually discerned. Research (including PET scans conducted while participants subitized or counted) suggests that subitizing is “a primitive cerebral process while counting involves more sophisticated operations” (Sousa, 2008, 14).

Researchers theorize that humans’ first experience with counting may have involved the fingers, and then moved to greater abstraction without the necessity of these manipulatives (Devlin, 2000, in Sousa, 2008). Other research confirms that fingers “can facilitate the transition between early nonverbal representations [of number] and conventional representations with number words” (Jordan & Levine, 2009, 64). Interestingly, when children are performing basic numeracy skills, the greatest brain activity occurs in the same motor cortex region of the brain that controls the fingers (Dehaene, Molko, Cohen & Wilson, 2004, in Sousa, 2008). “Finger-based number representations and finger-based calculation have deep roots in human ontology and phylogeny. Accumulating empirical evidence supporting the hypothesis of a neurofunctional link between fingers and numbers has emerged from both behavioural and brain imaging studies” (Kaufmann, 2008, 163). Current research has also found that finger gnosis (mental schema that recognizes and localizes fingers) and finger use are related to later calculation proficiency in elementary school children. "Finger-based counting and calculation may facilitate the establishment of mental number representations (possibly by fostering the mapping from concrete nonsymbolic to abstract symbolic number magnitudes), which in turn seem to be the foundations for successful arithmetic achievement." (Kaufmann, 2008, 163).

We can conclude that finger use may support and complement mental number representations as children learn to count and calculate. The development of number sense tied to counting and calculations does not end here, of course.

> "Although children between 3 and 6 years old have already begun formal instruction in mathematics, the present results show that the acuity of the ANS (Approximate Number System, also known as number sense) is still developing during this time. Indeed, the sharpening of the ANS does not appear to be complete until early adolescence. Given the central role this system plays in supporting mathematical intuitions, this protracted period of development highlights the importance of coming to understand the effects of changes in ANS acuity on math learning and achievement (Booth & Siegler, 2006; Jordan et al., 2007)." (Halberda & Feigenson, 2008, 1463)

The connection between the use of fingers and the development of number sense in young children validates Perry & Dockett's (2008) suggestion that these are the years when experiences can "hard-wire" associations in the brain that can then be further expanded.

The examination of early number sense and finger use with related executive functioning (see Blair et. al., 2008 and Simmons, Wills & Adams, 2012 for more information) is just one example from the research that is currently expanding our understandings of how young children naturally think mathematically and how we can thereby support children in further mathematizing those understandings over time.

We provide this one example of the cognitive processes underlying the development of number sense (subitizing, using fingers for counting and early mapping of these processes) as only a very small window into the vast knowledge that is emerging in regards to growth of young children's mathematics understanding.

Informal mathematics: What specific mathematics skills/concepts do children bring to school?

Certainly, the mathematics that children arrive with at school provides the foundation for learning more complex and abstract mathematics throughout school (National Mathematics Advisory Panel, 2008, in Jordan & Levine, 2009). Given that 80% of children can count and recognize shapes when they come to school, the fact that many preschool and kindergarten programs in North America are based on counting and recognition of shape, is incongruous. Researchers in this area, notably Sarama & Clements (2008), write that these observations argue for a raising of the curriculum ceiling, because "children often know more than curriculum developers or teachers give them credit" (68).

Despite strong universal starting points, there are striking differences in children's mathematical readiness by the time children enter preschool. As elaborated in the next section, these differences are apparent in number competency (e.g., determining set size, carrying out simple calculations, understanding and comparing quantities), shape recognition and measurement. Furthermore, they are apparent in mathematics language as well as non-verbal spatial sense (Zmich et al., 2011). This variance in children's mathematics readiness is highly effected by socio-economic status and other factors; a variance which the research demonstrates is related to later school success.

Section 2: The history of mathematics education for young children in Canada

In Canada, the story of mathematics education for young children has its roots in Europe and parallels closely the evolution of education for young children in the United States. It has been close to 180 years since the introduction of a publicly funded school program for young children – the first kindergarten program. Then, as now, debate about the most appropriate mathematics education for young children swings between two ends of a continuum. In this time, the education community has fluctuated between two opposing views: 1) that children are capable of and enjoy rich mathematical thinking; or 2) that early instruction in mathematics is unnecessary or even harmful to child development (Balfanz, 1999). In parallel, and further dividing the proponents of mathematics education for young children, is the argument over the most suitable and developmentally appropriate approach to mathematics instruction for young students. This argument oscillates between extremes of a teacher-directed or direct-instruction approach accompanied with the memorization of rules and algorithms, to a student- and/or play-centred approach allowing exploration with concrete objects, presumably leading to the discovery of patterns and rules along with deep conceptual understanding.

For Balfanz (1999), the limited depth in the kindergarten curriculum (the focus on numbers to ten and basic shapes) is the direct outcome of the war between these two camps that fought on an ideological basis during the late nineteenth and early twentieth century, when kindergarten was becoming institutionalized and standardized. In the first camp were educators such as Friedrich Froebel (the progenitor of kindergarten) and Maria Montessori (whose approach to education with an emphasis on manipulatives came out of her early experiences teaching students with special education needs and her realization that all students benefit from the techniques appropriate to teaching the student with special education needs) (Balfanz, 1999). Balfanz (1999) describes these educators as “naturalists” because their views of children’s capability in mathematics came from direct observation of children playing and learning in naturalistic settings. These educators proposed that young children could engage in serious intellectual work through play if they were provided with opportunities “to explore and discover fundamental mathematical properties in a prepared environment” (4).

In the other camp were social theorists who derived their views, according to Balfanz (1999), from theory rather than direct observation of children. In the early part of the twentieth century, it was the social theorists’ view, that early exposure to mathematics could be harmful, and this became the norm in the public education system. When kindergarten became a formalized component of the publicly funded education system in North America, the activities offered in kindergarten were linked to the activities and pedagogy of elementary school, and this led to the following outcomes: (1) Kindergarten was no longer included the “early years” of three to five, when the system offered only the year prior to Grade 1 (in order to fit within the elementary schools’ age-based system) (Balfanz, 1999); and (2) fundamentally, “kindergartens were transformed to fit with elementary schools” rather than designing kindergarten to meet the needs of the young children attending them. The implication of this history was that kindergarten came to be interpreted as a time of preparation of children to meet the expectations of the highly structured classrooms the children were to soon occupy. The formal incorporation of kindergarten into the public school system, according to Balfanz (1999), had a “particularly stultifying” effect with regard to mathematics education, “because the triumphant strands of elementary mathematics pedagogy during this era did not see value in challenging young children mathematically” (Balfanz, 1999, 8). What remains of Froebel’s early focus on mathematics in kindergarten is “largely forgotten or diluted” (Sarama & Clements, 2008). Interestingly, the approach of Maria Montessori can be found in the Montessori school system, where there continues to be explicit use of mathematics learning tools devised through her work with children.

Section 3: Development of children’s mathematical understandings: Socio-economic influences

Socioeconomic status (SES), which is defined as social position based on income, education and occupation (Lacour & Tissington, 2011) has been shown to be strongly linked to differences in mathematics competence. The gulf between middle and low-income children’s mathematics competence is wide. SES is associated with differences in cognition, achievement and behaviour as early as preschool (Duncan & Brooks-Gunn, 2000, in Baroody et al., 2006). And while there are known overall differences in preschoolers’ knowledge in many subjects, which are dependent on SES, they appear to be especially substantial in knowledge of mathematics (Case, Griffin & Kelly, 1999).

The correlation between low income and low mathematics achievement is well documented in the American research. There is ample evidence that a child’s SES is strongly predictive of mathematics achievement: kindergartners from low-income families lag far behind in mathematics knowledge of peers from more affluent families (e.g., Alexander & Entwisle, 1988; Geary, 1994, 2006; Jordan et al., 2007, 2009; Jordan & Levine, 2009).

Canadian context

In Canada, correlations between low income and low mathematics achievement have been identified. Although there are fewer studies that target mathematics outcomes, SES and other risk factors for school failure in Canada, there are findings from Ross & Roberts (1999) that reveal that children's problems with mathematics appear to decrease as family income rises.

A recent study by the Conference Board of Canada (2009), which examined child poverty situation nationally, ranked Canada 13 out of 17 Western countries to which it was compared. According to their analysis, more than one in seven children in Canada lives in poverty. (This organization uses the following definition of child poverty: “The proportion of children 17 years and under living in households where disposable income is less than half of the median in a given country.”) According to a study conducted by an anti-poverty advocacy group, the Canadian child poverty rate in 2008 was 15.2% (Campaign 2000, 2010). Statistics Canada puts this figure at 9.5% (Statistics Canada, 2009). The true child poverty rate likely falls between these two figures. (The difference in figures is attributed to differing definitions and measurements of poverty. See Canadian Council on Social Development’s 2001 position paper for more information.)

Starkey and Klein (2008) highlight the significance of sociocultural factors on the development of mathematical knowledge and skills in the early years. Specifically, the authors describe cross-cultural studies comparing mathematical abilities of economically advantaged and disadvantaged American and East Asian children and the amount of support for math learning they receive. Across all samples, it was demonstrated that economically disadvantaged children were provided with less support in their mathematical development compared to peers from economically advantaged backgrounds and as a result, fall behind academically. Although the learning gap appears to close in China by the second year of pre-school, it widens significantly in the American context.

As in the US, in Canada the income gap is also related to issues of race and gender equity. A recent study that looked at 2006 census data to compare the income of racialized Ontarians (self identified as a “visible minority” on census forms) to non-racialized Ontarians (Block, 2010) found a “striking difference” in income levels. Ontarians of race are “far more likely to live in poverty, to face barriers to Ontario’s workplaces, and even when they get a job, they are more likely to earn less than the rest of Ontarians” (3). Among the study’s Ontario statistics: for every dollar made by non-racialized men, racialized women made only 53.4 cents; for every dollar made by non-racialized men, racialized men made only 73.6 cents; for every dollar made by non-racialized women, racialized women made only 84.7 cents.

Poverty figures from Statistics Canada for 2008 (the most recent year in which figures were available at the time of writing) indicate that 1.6 million or 12.5% of people in Ontario lived in poverty (as measured by the Low Income Measure After-Tax, a measure commonly used in Canada that examines the proportion of family income required to cover the costs of basic necessities). According to the Canadian Council on Social Development (Maxwell, 2009), these statistics make Ontario the “child poverty capital of Canada.”

Parental input

Parental input into children’s mathematical experiences and learning is critical, as it has been shown that “parental social class and education level predicts mathematics achievement throughout elementary and secondary school” (Jordan & Levine, 2009). Levine et al. (2010) found that socioeconomic status (SES) was a key factor that predicted both the frequency and complexity of the number activities reported (such as counting, number-object matching, and use of number words). There is a small but robust literature on the influence of home environments on children’s mathematics abilities. Blevins-Knabe & Musun Miller (1996) found a correlation between numeracy activities at home (such as verbal counting) and children’s performance on a standardized test. LeFevre et al. (2009) found a correlation between children’s mathematics performance and their parents’ reports of home numeracy activities.

Social emotional factors

Indeed there has been a plethora of studies demonstrating that the results vary greatly from child to child in terms of mathematics performance in the early years. Many theories have been put forth as to the factors associated with these differences. Stipek & Ryan (1997) were interested in investigating the social emotional factors. They hypothesized that motivation for learning mathematics would have an impact on children’s mathematics performance.

In Stipek & Ryan’s (1997) much-cited study, 233 preschool children were assessed in the fall and spring on eight cognitive tests. Results revealed much poorer performance on all tests for lower SES children. The gains in both groups over the year were roughly equal and differences between groups remained approximately the same. Motivation, on the other hand, did not fluctuate in children - regardless of their SES. The researchers found that SES-related cognitive differences in children were substantial and that kindergartners from economically disadvantaged backgrounds scored lower on the cognitive measures than advantaged preschoolers. Furthermore, out of all of the SES variables, income was the most consistent predictor of the children's cognitive abilities. The findings from their study underscore the importance of establishing effective early childhood programming to serve children of all SES levels.

Development of number sense and early arithmetic

In particular, research has focused on development of number sense and early arithmetic. These investigations have revealed differences on a wide range of foundational tasks: recognizing written numerals, reciting the counting string, counting sets of objects, counting up or down from a given number other than 1, adding and subtracting, and comparing numerical magnitudes (Ginsburg & Russell, 1981; Griffin, Case & Siegler, 1994; Jordan et al., 1992; Jordan, Kaplan, Olah & Locuniak, 2006; Saxe, Guberman & Gearhart, 1987; Starkey, Klein & Wakeley, 2004; Stipek & Ryan, 1997). In all of these studies the consistent findings are that low SES students lag behind their middle class counterparts. These differences are significant because early number knowledge is highly predictive of mathematics performance in later grades.

Development of spatial reasoning

While there has been a great focus on children's numerical development in early years, inadequate attention has been paid to what *preschoolers* know about geometry (Clements & Sarama, 2009, 2011).

> Spatial thinking is an essential human ability that contributes to mathematical ability. It is a process that is distinct from verbal reasoning (Shepard and Cooper 1982) and functions in distinct areas of the brain (Newcombe and Huttenlocher 2000). Further, mathematics achievement is related to spatial abilities (e.g., Ansari et al. 2003). As an example, empirical evidence indicates that spatial imagery reflects not just general intelligence but also a specific ability that is highly related to ability to solve mathematical problems, especially nonroutine problems (e.g., Wheatley et al. 1994). This is particularly important because some individuals are harmed in their progression in mathematics due to lack of attention to spatial skills, benefit from more geometry and spatial skills education (e.g., Casey and Erkut 2005). (Clements & Sarama, 2011, 134)

A study by Zmich et al. (2011) was designed to discover what 3-year-olds know about assembling geometric forms in two dimensions. The primary purpose of this study was to investigate gender differences in children at this age. In this study, the children were shown a design composed of various shapes and were then given the same individual shape pieces and asked to make their pieces look just like the model design. Results showed no gender differences in the children's abilities to copy the target shapes. What was revealed, however, was that even at this early age, (the participants were 3.8 years old on average) low-income children performed worse than the middle class children in the study. The results were particularly surprising because the task that the children were asked to was essentially non-verbal and many of the lags of low SES children in early mathematics development are directly related to language development.

Intervention: An equity issue

While socio-cultural factors certainly impact children's early mathematical development, "cross-cultural and intervention studies indicate that when preschool children are provided with high quality mathematics experiences, they can eliminate the socio-economic gap in their mathematical knowledge" (Starkey & Klein, 2008, 253). High quality early instruction in mathematics is critical to provide equitable opportunities for achievement and learning for *all* children.

For at-risk children, early intervention and education, in the form of rich mathematical experiences, provide the only hope of closing the gap created by SES differences – a gap that widens without intentional mathematics intervention. The following studies sample the research in this area.

- Jordan et al. (2007) found that number competence in kindergarten strongly predicted the rate of growth between grades 1 and 3 as well as achievement level in grade 3, and that this was a stronger indicator than SES. "These data suggest that number competence, which can be taught and learned, could be a key factor in bridging the income gap in mathematics achievement" (64).

- In a later study, Jordan & Levine (2009) argue that early intervention is essential for helping all learners and that without such help, learners of low SES and/or with mathematics difficulties, will experience "a cascade of mathematics failure" that threatens their ability to ever catch up to their mathematically advantaged peers (6). Jordan & Levine (2009) point out that the "consequences of poor mathematics achievement are serious for daily functioning and for career advancement," (60) because success in mathematics is associated with entry into higher learning and occupations in sciences, technology, engineering and mathematics. Low success in math is limiting for later opportunities in life and can contribute to the continuation of cycles of poverty. Jordan & Levine (2009) examine the characteristics of mathematics learning difficulties in elementary school, among them poor calculation fluency, which can be diagnosed and addressed early in school to avoid ongoing negative impacts on achievement. According to the study, "most children with mathematics difficulties in first grade and later seem to have particular problems with the verbal or symbolic systems of number, which are heavily influenced by early experiences and instruction" (62).

- Baroody, Bajwa & Eiland (2009) found that the primary cause of problems with basic number combinations among young children "is the lack of opportunities to develop number sense during the preschool and early school years" (69).

- Heckman (2011) analysed many US based studies, reaching the conclusion that non-academic skills are also important when considering interventions that close gaps. He outlines four key points:

  > 1. Inequality in early childhood experiences and learning produces inequality in ability, achievement, health, and adult success.
  > 2. While important, cognitive abilities alone are not as powerful as a package of cognitive skills and social skills—defined as attentiveness, perseverance, impulse control, and sociability. Cognition and personality drive education and life success, with character (personality) development an important and neglected factor.
  > 3. Adverse impacts of genetic, parental, and environmental resources can be overturned through investments in quality early childhood education that provide children and their parents with the resources they need to properly develop the needed cognitive and personality skills.
  > 4. Investment in early education for disadvantaged children from birth to age 5 helps reduce the achievement gap, reduce the need for special education, increase the likelihood of healthier lifestyles, lower the crime rate, and reduce overall social costs. Every dollar invested in high-quality early childhood education produces a 7% to 10% per annum return on investment. Policies that provide early childhood educational resources to the most disadvantaged children produce greater social and economic equity. (5; See also Cunha & Heckman, 2010.)

The research is clear that, in spite of the mathematical assets that children possess, the informal knowledge that kindergarten-age children bring to their formal schooling experience varies widely among social class and other categories (Baroody et al., 2006). Aside from individual differences explained by genetic or acquired impairment, "most individual differences are probably due to the lack of opportunity" (Baroody et al., 2006, 200). A focus on establishing the foundations of mathematics knowledge early in formal schooling, then, "seems to be an essential first step for achieving *equity*. …Early intervention is now viewed as one key step toward ensuring a level playing field" (Baroody et al., 2006, 202).

According to Baroody, Lai & Mix (2006), informal mathematics knowledge forms the basis for understanding the formal mathematics taught in school. Formal instruction, then, should build on and connect to informal knowledge and experiences; "gaps in informal knowledge need to be identified and filled early (during the preschool years or the first years of school)" (199). However, in many cases, this gap closing is not happening.

Section 4: Impact of mathematics on children's later learning: Mathematics as a predictor of later achievement

Starting school on solid footing supports positive outcomes for students, the education community and the wider community in the long-term. Research tells us that young children's mathematics knowledge and abilities play a critical role in long-term school success.

A key area of research on young children and mathematics stems from studies based in several different countries including the United States and Canada. In 2007, Duncan et al. analysed six longitudinal data sets to discover that mathematics skills at kindergarten entry were the *best* predictor of later school achievement, and that this pattern was consistent for both boys and girls from both high and low socioeconomic backgrounds. The 2007 study demonstrated that "early math is a more powerful predictor of later reading achievement than early reading is of later math achievement" (1443). This 2007 study is considered by many to be the largest (n=36,000) and most comprehensive longitudinal study on school readiness, establishing that the cornerstones to school readiness were informal mathematics skills, including knowledge of the number line and ordinality.

To further their work, in 2009, Claessens, Duncan & Engel, used the American Early Childhood Longitudinal Study-Kindergarten Cohort (ECLS-K), a nationally representative database of 21,260 children who started kindergarten in 1998-1999. This database had, at the time of the study, produced five waves of data: Fall of Kindergarten, Spring of Kindergarten, as well as grades 1, 3 and 5. The researchers examined the outcomes of the 11,820 respondents still in the study in grade 5 to determine which skills – literacy, numeracy or socio-emotional and attention skills – were most predictive of success in grade 5. Their intention was to pinpoint areas to target for preschool interventions. (By extension, it seems reasonable that similar interventions in the kindergarten year and beyond would also be important.) Their main finding again confirmed that *children's school-entry mathematics abilities were not only consistently predictive of later achievement in mathematics but in reading as well*: "[r]udimentary math skills were the single most important set of kindergarten-entry skills emerging from our analyses, followed by reading skills, and finally attention skills, which were consistently predictive of both math and reading outcomes" (423).

Building on this work, Claessens & Engel (2011) wanted to determine which specific mathematics knowledge and skills were predictive of future academic success: "[g]iven that theory and research indicate that by age 5 children have processing skills that will allow them to learn a range of mathematics skills, understanding which skills are most beneficial for later school outcomes has important implications for early mathematics education" (4). Again using the ECLS-K database, including a variety of achievement tests and teacher/parent questionnaires, the researchers examined the specific early mathematics knowledge and skills children need for success in elementary school through to middle school. They found that the subscale which measured a "child's ability to read all one-digit numerals, count beyond ten, recognize a sequence of patterns and use nonstandard units of length to compare objects is typically the most consistent and important predictor of later achievement test scores in both reading and math across elementary school" (13).

The researchers also investigated how achievement in early mathematics influences children's academic outcomes across different subject areas and its predictability of grade retention for both advantaged and disadvantaged children. They found that "early math skills are important for a broad range of measures of school success including reading, science, and grade retention" (13) and that this held true across SES. Jordan & Levine (2009) agreed that children's number sense upon school entry sets the foundation for later learning of more complex mathematics.

The most recent in this string of related research studies, a study by Watts, Duncan, Siegler and Davis-Kean (2014) tracked 1364 children from 54 months of age to age 15. The results were startling; the best predictor of mathematics performance at age 15 was *not* the children's initial mathematical understanding as measured at 54 months, but the extent of their *growth* in mathematical understanding from Kindergarten to the end of Grade 1. Children's gains in understanding from approximately 4.5 to 7 years of age – in other words, the mathematics learning that children do in that time period – fertilizes the ground for their success at age 15. This research underlines the critical importance of providing young children with explicit learning opportunities in mathematics.

The Case of Spatial Reasoning

Clements and Sarama (2011) argue that geometry, the strand of math most directly connected to spatial reasoning, should be of the highest priority because it too predicts later school achievement. "Empirical evidence indicates that spatial imagery reflects not just general intelligence but also a specific ability that is highly related to (the) ability to solve mathematical problems, especially non-routine problems" (134). (See also Wheatley et al., 1994). This is particularly important because some individuals are harmed in their progression in mathematics due to lack of attention to spatial skills (Casey and Erkut, 2005; Clements & Sarama, 2011).

What is spatial reasoning?

Spatial reasoning refers to our capacity to relate to and navigate the wider world around us, and involves the ability to create and mentally manipulate "representations of actual and imagined shapes, objects, and structures" (Cohen & Hegarty, 2012, p. 868). *Spatial reasoning* is not limited to a single ability or process, but refers to a variety of skills and concepts, as well as the tools used to represent and communicate ideas about space and spatial relationships (National Research Council, 2006). The Spatial Reasoning Study Group, a think tank of mathematicians, math educators, and psychologists from Canada and the United States, has developed a list of actions involved in spatial reasoning. These include (but are not limited to) perspective taking, visualizing, locating, orienting, dimension shifting, pathfinding, sliding, rotating, reflecting, diagramming, modelling, symmetrizing, composing, decomposing, scaling, map-making, and designing (Davis, Okamoto, & Whiteley, 2015).

Spatial thinking is important in all areas of mathematics and beyond; most subjects in school—art, geography, science, language, and physical education to name a few—rely on at least some aspects of spatial thinking. Converging evidence from the psychology research literature has revealed that people who perform well on measures of spatial ability also tend to perform well on measures of mathematics performance and tend to be better problem solvers (Mix & Cheng, 2012) and are more likely to enter into and succeed in STEM (science, technology, engineering and math) disciplines (Wai, Lubinski, & Benbow, 2009). In fact, the relationship has been so strongly demonstrated in the research that Mix and Cheng (2012) put it this way: “The relation between spatial ability and mathematics is so well established that it no longer makes sense to ask whether they are related” (p. 206). Further evidence for the link between spatial thinking and mathematics comes from studies showing overlapping brain regions involved in both activities (Geary, 2007). Researchers have noted that the link between spatial reasoning and math is so strong, it is “almost as if they are one and the same thing” (Dehaene, 1997, p. 125). Reflecting on the strength of this relationship, researchers have predicted that “spatial instruction will have a two-for-one effect” that yields benefits in mathematics as well as the spatial domain” (Verdine et al., 2013, p. 13).

In one of the first studies of its kind to show *specific* links between spatial and math skills, Cheng and Mix (2013) conducted a study in which children were assessed in both spatial and math skills, and then randomly assigned to one of two groups. The first group engaged in activities that involved performing mental rotations (an area of spatial reasoning shown to be strongly linked to mathematics skills); the second group worked on crossword puzzles for an equivalent length of time. Post-test results showed that the children in mental rotation group outperformed the crossword puzzle group in a variety of measures of spatial reasoning and mathematics. In particular, those children demonstrated significant improvements in assessments of calculation skills, specifically in missing term problems (e.g., 5 + ___ = 7). The researchers speculate that this might be because the children were better able to mentally manipulate the numbers after practice with mental rotations.

Spatial reasoning as a predictor of mathematics success

Notably, very recent studies are now identifying spatial reasoning as a key that opens many doors for children. There is good evidence that spatial reasoning experiences at an early age contribute to children's "development of both numerical and spatial/geometrical concepts" (National Research Council, 2009, 184; see also Casey & Erkut, 2005; Starkey, Klein, & Wakeley, 2004). A study by Gunderson et al. (2012), for example, found that children who have strong spatial skills at age 5 tended to perform well on number line tasks in Grades 1 and 2. Research has shown that the ability to represent numbers on a number line is an important skill that is related to the ability to conduct operations with numbers; these findings suggest that possession of a 'mental number line' contributes to children's future success in mathematics (Siegler & Booth, 2004; Booth & Siegler, 2006).

In another study, Verdine, Irwin, Golinkoff, and Hirsh-Pasek (2014) found that a child's spatial skills at age 3 reliably predicted their number knowledge, including their attainment of number concepts such as *more*, *less*, *equal*, and *second*. The researchers also found that activities such as copying 2D and 3D shapes could potentially improve children's spatial skills. Researchers have also found that children's spatial reasoning ability at age 3 was a stronger predictor off their mathematical (arithmetic) skills at age 5 than either their vocabulary, and perhaps most surprisingly, their math performance at age 3 (Farmer et al., 2013). These findings suggest that, not only is mathematics important in the early years, but even more precisely targeted intervention and experiences in spatial reasoning have enormous potential for improving children's later mathematics learning and success.

Lack of attention to spatial reasoning

Yet spatial reasoning itself rarely, if ever, receives explicit attention in curricula. Unfortunately, the development of mathematics understanding in some children is hindered due to this lack of attention to spatial skills (Casey & Erkut, 2005; Clements & Sarama, 2011). The National Council of Teachers of Mathematics recommends that at least 50 percent of mathematics instruction focus on a spatial approach (National Council of Teachers of Mathematics [NCTM], 2006, 2010). Despite calls to bring geometry and spatial thinking to the forefront of early math curricula, local and international studies reveal that geometry and spatial sense receive the *least* amount of attention in early years math (Bruce, Flynn, & Moss, 2012; Sarama & Clements, 2009), making it an underserved area of mathematics instruction. While the relationship between mathematics, science, and spatial reasoning has been at least partially understood for decades in the world of psychology, this knowledge has not had an impact on the design of school curriculum until recently (Mix & Cheng, 2012; National Research Council, 2006). As Newcombe, a world expert in spatial reasoning, asserts: "spatial reasoning is like the 'orphan' of the academic curriculum and has never been a focus of instruction, as are reading, writing and arithmetic" (Newcombe, Uttal & Sauter, 2013, 45). The National Research Council (2006) has highlighted this as a "major blind spot" in education and calls on educators and researchers to pay attention to spatial reasoning. Otherwise, the Council warns, spatial reasoning “will remain locked in a curious educational twilight zone: extensively relied on across the K–12 curriculum but not explicitly and systematically instructed in any part of the curriculum” (p. 7). Geometry and spatial reasoning in the early years typically focus on having children label and sort shapes (Clements, 2004), yet cognitive science and educational research, including research conducted by Bruce, Flynn and others in Ontario since 2011, shows us that young children are capable of—and interested in—more dynamic and complex spatial thinking (see Moss, Bruce, Caswell, Flynn & Hawes, 2016; Bruce, Flynn & Bennett, 2015; Moss, Hawes, Caswell, Naqvi, Bruce & Flynn, 2014; Bruce, Moss & Flynn, 2013).

Fortunately, new research is starting to shine a spotlight on the importance of spatial reasoning in mathematics learning, and is gaining some attention in Ontario at the Ministry and school levels. In 2014, the Ontario Ministry of Education published a curriculum support document, Paying Attention to Spatial Reasoning (Flynn & Hawes, 2014), and sent a copy to every school in Ontario. As this is an emerging area of research and learning, more needs to be understood about how spatial reasoning benefits students mathematics learning, and about how to implement this approach to learning in classrooms. Furthermore, support for educator learning in this area is needed; not only to bring awareness of what spatial reasoning is and its importance, but to provide guidance on how to support student spatial reasoning to foster children’s mathematical development.

Section 5: The state of mathematics education for young children today: Research on educator values, practices and challenges

Values

Helping children develop socially and emotionally (Kowalski, Pretti-Frontczak & Johnson, 2001; Lee, 2006; Lin, Lawrence, & Gorrell, 2003) and protecting them from tedium and stress are reported as top priorities for educators of young children (Lee, 2006; Lee & Ginsburg, 2007b). According to Lee & Ginsburg (2007a), mathematics learning is not a top priority for most educators of young children. In their research, they have also revealed that many educators believe that their students will eventually catch up to their peers mathematically, regardless of what happens during their early childhood (Lee & Ginsburg, 2007a, 2007b).

Studies also indicate that educators of young children favour teaching mathematics in a child-centred environment (Lee, 2005; Lee, 2006; Jung & Reifel, 2011; Wang et al., 2008). In particular, educators of young children believe that play-based learning is important. Generally, educators report implementing a play-based learning environment by providing a rich physical environment that contains appropriate mathematics materials, manipulatives, and activities (Lee, 2005) and then allowing children the opportunity to choose which activities to engage in (Lee, 2006). They believe that children learn a great deal through self-discovery (Lee, 2006) and this belief in the importance of play is reflected in Early et al.'s (2005) finding that preschool students spend 28% of the day in free choice and center-based activities.

One serious concern related to mathematics education for young children is that educators may select a career in educating young children (rather than older students) *because* they do not like mathematics or they do not want to teach mathematics (Ginsburg & Ertle, 2008). Ginsburg & Ertle (2008) describe this at least in part as an identity issue: many do not see themselves as teachers of mathematics and indeed, report that they entered the profession to avoid teaching mathematics because they dislike it or have been unsuccessful in mathematics themselves. Compared to other elementary educators, kindergarten to grade 4 teachers have been found to have the strongest negative attitudes toward mathematics (Kolstad & Hughes, 1994). In 2004, Copley found that early childhood educators generally felt most comfortable teaching literacy and language skills, while considering mathematics to be difficult to teach. This concurs with a study conducted by Lee and Ginsburg (2007b), in which substantial interview data (combined with written vignettes from 60 preschool teachers) showed that educators had different pedagogical beliefs and practices for mathematics teaching compared to literacy (for example, incorporating mathematics into classroom routines, whereas they described literacy as the core of their program): "overall I'm feeling I don't know much about teaching math. I know a little bit, you know, enough that, I know which materials to provide the children" (teacher quote, 134). Ginsburg & Ertle (2008) note that a lack of knowledge may underlie these difficulties, as teaching early mathematics concepts effectively requires a great deal of mathematics knowledge. In fact, in a survey of 384 primarily female pre-service elementary teachers, Perry (2011) found that those who chose careers in elementary education reported low confidence levels in learning mathematics, lower, on average, than women who chose careers in other fields. Ginsburg, Lee & Boyd (2008) have found that "…in general, early childhood teachers do not place a high value on teaching mathematics" (10).

Practices

Even when educators do value mathematics instruction for young children (as has been found among educators in low-SES populations), Brown (2005) did not find a positive relationship between educators of young children's beliefs about the importance of mathematics and their actual practices. Educators who have difficulty putting their beliefs into action may be responding to outside social influences such as government policy (Jung & Reifel, 2011; Wang et al., 2008), professional training (Wang et al., 2008), parents (Charlesworth, Hart, Burts & Hernandez 1991; Lee & Ginsburg, 2007a), other educators (Jung & Reifel, 2011), as well as pressures to achieve high standardized test scores (Jung & Reifel, 2011). Also, these educators may be constrained by factors within their classroom, such as the presence of many exceptional learners (Dunphy, 2009) or personal factors such as a lack of mathematics knowledge (Ginsburg & Ertle, 2008).

American studies have found that, on average, educators of young children engage in mathematics activities for a small portion of the day (7-17%), a smaller proportion than both literacy (18-30%) and social studies activities (13-24%) (Phillips, Gormley & Lowenstein, 2009).

Research conducted in Ontario shows similar trends. For example, in 2011, a study was conducted with 631 Ontario teachers of Junior Kindergarten to Grade 2 regarding mathematics teaching practices. Teachers reported spending substantially more time on literacy than on mathematics: 37.5% of respondents reported spending more than 30% of their time on mathematics tasks whereas 85.5% of respondents reported spending more than 30% of their time on literacy tasks (results from 2012 Ontario survey: Bruce, Ross & Moss). In this same survey, Ontario teachers reported that when they do engage students in mathematics learning tasks, they spend the least amount of time on geometry and spatial reasoning tasks (ranked 4th out of 5 areas in Kindergarten and ranked 5th out of 5 areas in Grades 1-2). These findings are supported by other US studies such as Lee (2010) who found that 81 teachers of young children demonstrated a significantly higher level of pedagogical content knowledge in number sense compared to all other content areas and that the lowest teacher scores were in the area of spatial sense. Yet, spatial reasoning experiences at an early age contribute to children's "development of both numerical and spatial/geometrical concepts" (National Research Council, 2009, 184).

To further complicate matters, some educators and policy makers believe that mathematics learning for young children should come from children's play alone. Paradoxically, most children already come to school with a baseline of mathematics understanding that is largely ignored. Given that 80% of children can count and recognize shapes when they come to school, the fact that many preschool and kindergarten programs in North America are based on counting and recognition of shape is incongruous. Researchers in this area, notably Sarama & Clements (2008), write that these observations argue for a raising of the curriculum ceiling, because "children often know more than curriculum developers or teachers give them credit" (68).

An exclusively play-based environment tends to lean towards the "teachable moment" and capitalizes on the emergent mathematical behaviour and talk of children at play. Unfortunately, capitalizing on teachable moments, when they are recognized, is "extraordinarily difficult" (Ginsburg & Ertle, 2008, 47). Ginsburg & Ertle's (2008) extensive observation of classrooms "suggest that teachers seldom exploit the mathematics in children's everyday behaviour" (60). Teacher decision making and the ability to capitalize on teachable moments requires deep mathematical knowledge (Ginsburg & Ertle, 2008) and fortuitous opportunity.

Challenges

Perry and Dockett (2008) describe "one of the tensions in mathematics teaching and learning in the early childhood years:"

> While children demonstrate remarkable facility with many aspects of mathematics, many early childhood teachers do not have a strong mathematical background. At this time when children's mathematical potential is great, it is imperative that early childhood teachers have the competence and confidence to engage meaningfully with both the children and their mathematics. (99)

This is of great concern because recent studies show "that teacher knowledge is significantly correlated with student achievement in grades 1 and 3" (Hill, Rowan & Ball 2005).

Skipper & Collins (2003) help to elucidate the challenges for educators of young children in finding an approach to teaching that capitalizes on children's informal mathematics and natural curiosity through play, while providing purposeful learning experiences:

> Teachers whose formal educations did not cover specific strategies to build on young children's intuitive understandings of mathematical concepts may make two kinds of mistakes. Some may fall back on a general concept that play is the only important and developmentally appropriate approach for young children. These teachers may favour a completely unstructured approach in which mathematical learning is believed to occur incidentally during play, with little teacher participation. Other teachers with less formal training may rely on their own understandings of what it means to be a teacher, perhaps by imitating the

> way they were taught in elementary school. This group may be more comfortable with a highly structured or scripted approach. *Neither of these approaches maximizes opportunities for young children to connect mathematical concepts to the real world in meaningful ways.* (421-422; emphasis added)

As a result of limited understandings of children's mathematical capacities, some educators hold the belief that children can only think in concrete terms and that abstract ideas should be avoided. Another misconception educators may hold is that students are not ready for greater breadth and depth of mathematics material (Lee & Ginsburg, 2007b). Some believe that integrating mathematics into other subject areas or into every day routines (e.g., taking attendance using tallies) is sufficient (Lee & Ginsburg, 2007a; Lee & Ginsburg, 2009; Wang, Elicker, McMullen & Mao, 2008). In contrast, in the same interviews, educators indicated that they believed it was important to set aside time to focus specifically on literacy (Lee & Ginsburg, 2007a).

These challenges must not be interpreted as a criticism of educators, but of systematic, cultural and historical factors. As Stipek (2008) cogently writes:

> We cannot blame the teachers. Until recently we have not expected instruction in mathematics in early childhood education programs. And in addition to not being trained, many are not comfortable with their own mathematical skill. Furthermore, the difficulty of teaching young children mathematics is typically underestimated (in,Ginsburg, Lee & Boyd, 2008, 13).

Ginsburg & Ertle (2008) echo this concern, and ask how, with limited training in mathematics, should teachers "then be expected to understand something as complex as early mathematical ideas or children's mathematical thinking" (62)? These concerns point to an urgent need to support continual improvement of the quality of teacher education programs by supporting teachers in their own mathematics learning and in their learning of sound mathematics pedagogy for young children, through more rigorous mathematics content in pre-service education, as well as extensive, frequent and long-term in-service learning support for practicing teachers. Baroody, Lai & Mix (2006) argue for pre- and in-service opportunities for teachers that include integrated mathematical content, as well as a focus on conceptual understanding and how to address "developmentally appropriate big ideas" (211).

Ginsburg, Lee & Boyd (2006) argue that because of historical and social contexts and the limited nature of mathematics teaching in the early years, the field doesn't yet have a full understanding of what children are capable of in mathematics. They suggest that views of high quality mathematics education for young children should be supported by research in atypical teaching and learning situations without setting limits on what children can do. "We need to conduct teaching experiments that provide unusually stimulating conditions designed to push children's performance and learning to their outer limits. Before the web's invention, we could not have known that 4-year-olds could surf it" (16).

Although in recent years trajectories of early learning have been established by the likes of Clements and Sarama (2009), there is still a long way to go in understanding the way very young children learn mathematics. Another challenge has to do with effective training of teacher candidates and the kinds of professional learning that is offered for teachers of young children, which is still emerging.

Section 6: What could mathematics education for young children look like? Lessons from the research

Importance of play

Perry & Dockett (2008) stress the importance of play in the mathematical development of young children. Bergen (2009) describes play as a "medium for learning" that provides opportunities for communicating (even before verbal skills are fully developed), risk taking, confidence building, as well as for developing self-regulation and social skills (416). Play, including imaginative pretense, construction play, and games with rules, promote and enhance logico-mathematical reasoning as well as social understanding and metacognition (Bergen, 2009). As a disposition, play is closely linked to other characteristics valued in mathematics education, including creativity, curiosity, problem posing and problem solving (Ginsburg, 2006; NAEYC/NCTM, 2002; Dockett & Perry, 2007).

Free play

Sarama & Clements (2008) describe the *potential* mathematics in children's free play and demonstrate, through observational data, children's natural interest in these potentially mathematical situations, including classification, magnitude, enumeration, dynamics, patterns and shape as well as spatial relations. Hunting (2007) describes children's mathematical play as "big play," defined as "self-motivated and self-directed activity that… features embryonic mathematical thinking, which, in the estimate of the astute teacher… may present an opportunity for conversation, discussion, a question, or just observation and recording for later investigation" (729).

Ginsburg (2006) also discusses the possible mathematics in children's spontaneous play, including an interest in relative distance (how close or how far), relative magnitudes (how much or how many), location of objects, quantity (where children show an interest in counting to high numbers such as 100) as well as measurement and patterns. In a study where investigators videotaped 90 children (4-5 years of age) drawn from low, middle and high SES populations for 15 minutes each day to observe their "everyday mathematical behaviour", researchers coded three types of mathematical behaviour that frequently occurred (Seo & Ginsburg, 2004). The researchers found that pattern and shape explorations occurred an average of 21% of the 15 minutes, magnitude explorations occurred an average of 13% of the 15 minutes, and enumeration occurred an average of 12% of the 15 minutes. Interestingly, there were no significant differences among the SES categories. These findings bear implications for classroom programming for young children: "if children explore these topics on their own, there is good reason to include them in the curriculum…the curriculum can be much more challenging than it is now" (158).

Further, these findings are promising in that they imply that SES gaps can be addressed with early intervention and teaching at school entry. Ginsburg, Lee &

Boyd (2008) clarify the situation when they write that "lower-SES children exhibit difficulty with *verbal* addition and subtraction problems, they perform as well as middle-SES children on *non-verbal* forms of these tasks" (5). However, the fact that they do not differ in their expressions of mathematical ideas in spontaneous free play suggests that tasks relating to non-verbal number competence might allow educators an entry point to build on these students' strength.

Playful mathematics and learning through play

> *Playful math is what some call "pure math." It is what real mathematicians do, and it is also what 4-year-olds do. Playful math is to numbers what poetry is to words, or what music is to sounds, or what art is to visual perception. … Playful math involves the discovery or production of patterns in numbers, just as poetry involves the discovery or production of patterns in words, and music involves the discovery or production of patterns in sounds, and art involves the discovery or production of patterns in visual space. (Gray, 2010)*

Play has long been accepted as a legitimate mode of learning for young children – as a way that children incorporate new information, learn to socialize, practice important verbal and kinesthetic skills, and problem solve. Perry & Dockett (2008) describe play as a particularly important part of children's transition to school (where demands on children increase, as support such as the ratio of teacher-to-student is reduced). Play also has a role in bridging prior-to-school experiences to within-school experiences (bringing increased comfort and confidence in the new environment, for example) (Perry & Dockett, 2008). Perry & Dockett (2008) stress the importance of play in the mathematical development of young children.

How is play defined? In addition to providing some element of fun, some additional defining characteristics include internal control (relating to choice), internal motivation (relating to engagement) and internal reality (the freedom to shape the "reality" of the activity). Cognitive, physical and social spontaneity are also components of playful activity. Researchers have discovered links between play and divergent thinking, verbal intelligence, creative problem solving, cognitive flexibility and adaptation to change (Bergen, 2009). Bergen (2009) describes play as a "medium for learning" that provides opportunities for communicating (even before verbal skills are fully developed), risk taking, confidence building, as well as for developing self-regulation and social skills (416). All forms of play, including imaginative pretense, construction play, and games with rules, promote and enhance logico-mathematical reasoning as well as social understanding and metacognition (Bergen, 2009). Bergen (2009) links children's play to the creative and playful thinking in which accomplished mathematicians and scientists engage. Current research focuses on play as a process (involving non-literal ways of thinking, multiple possible outcomes, and no one "right answer") as well as a disposition or habit of mind (involving an attitude towards objects and ideas) (Dockett & Perry, 2007). As a disposition, play is closely linked to other characteristics valued in mathematics education, including creativity, curiosity, problem posing and problem solving (Ginsburg, 2006; NAEYC/NCTM, 2002, in Dockett & Perry, 2007).

Those participating in block play, model building, carpentry or playing with art materials do better in spatial visualization, visual-motor coordination, and creative use of visual materials (e.g., Caldera, McDonald-Culp, Truglio, Alvarez & Huston, 1999; Hirsch, 1996; Wolfgang et al., 2001). A wealth of empirical data also shows that teachers can enrich learning through children's play by adding math- and literacy-related materials into school environments (e.g., Christie & Enz, 1992; Christie & Roskos, 2006; Arnold, Fisher, Doctoroff & Dobbs, 2002; Griffin & Case, 1996; Griffin, Case & Siegler, 1994; Einarsdottir, 2005; Kavanaugh & Engel, 1998; Roskos & Christie, 2004; Saracho & Spodek, 2006; Stone & Christie, 1996; Whyte & Bull, 2008). For example, Cook (2000) found preschool children engaged in more talk and activities relating to mathematical concepts when number symbols were embedded within play settings. Another research study found that the quality of block play at four years of age was a predictor of high school mathematics achievement (Wolfgang et al., 2001). Another study found a relationship between young children's construction skills (e.g., playing with jigsaw puzzles, blocks and Lego) and strong number sense and performance in mathematical word problem solving (Nath & Szücs, 2014; Oostermeijer, Boonen, & Jolles, 2014).

It should be noted that the consensus in the literature is that "play does not guarantee mathematical development, but it offers rich possibilities. *Significant benefits are more likely when teachers follow up by engaging children in reflecting on and representing the mathematical ideas that have emerged in their play*" (NAYCM/NCTM, 2002, 10; emphasis added). As deVries, Thomas & Warren (2010) write, "play is a pedagogical tool that can enable learning and this learning can be maximized with appropriate, timely and effective adult input" (719). There continues to be room for learning what playful mathematics looks like and feels like, to expand our understanding of how children play with mathematical ideas.

Play is not enough

Although it is without question that children learn through play, current research indicates that a reliance on the teachable moment presents challenges and will not be enough to provide quality learning experiences for children in mathematics. Ginsburg, Lee & Boyd (2008) note that "there is good reason to believe that in practice the teachable moment is not an effective educational method" (7), because teachers may have little time to spend in careful observation required to attend to such moments. Further, Ginsburg, Lee & Boyd (2008) question whether the mathematical content knowledge of many teachers of young children is sufficient to recognize the mathematics embedded in young children's play and to be able to plan enriching learning activities, especially in the context of classrooms with 20 children from diverse backgrounds. Intentional teaching must be a "key part" of early childhood mathematics education; Ginsburg, Lee & Boyd (2008) do not mince words when they write that "it is the responsibility of educators to do more than let children play or respond to teachable moments" (8), but that teachers must provide a variety of experiences for children to intentionally foster the development of mathematical concepts, skills and language.

Indeed, according to the position statement given by the National Association for the Education of Young Children (NAEYC) and National Council of Teachers of Mathematics (NCTM) (2002), educators need to consider *at least* two approaches towards the mathematics education of young children: 1) maximizing the opportunities provided by the "teachable moment" in children's play to build mathematical ideas; and, 2) enacting an intentional curriculum designed to sequence mathematical ideas in a developmentally appropriate manner. The policy explicitly states that mathematics in the early years "needs to go beyond sporadic, hit-or-miss mathematics," but needs to provide "carefully planned experiences that focus children's attention on a particular mathematical idea or set of related ideas" (NAEYC and NCTM, 2002, principle 9).

When examining the play literature, it is tempting to juxtapose play and instruction. However, these are not mutually exclusive categories or approaches – both provide learning opportunities and in fact the overlap is natural and desirable: "high quality instruction in mathematics; and high quality free play need not compete for time in the classroom. Engaging in both makes each richer and children benefit in every way" (Sarama & Clements, 2009, 331). Baroody, Lai & Mix (2006) concur with the conclusions of the NAEYC and the NCTM when they note that "it is doubtful that incidental experiences or learning will promote educative experiences… [and that] existing evidence indicates young children are ready for organized, sequenced experiences embedded in specific activities …or *a careful combination of approaches*" (204; emphasis added).

Instructional strategies for engaging young children in mathematics – the importance of explicit teaching

As noted by Balfanz (1999) and others, the mathematical education of young children has never been without controversy; one of the points of contention is whether direct instruction in mathematics is appropriate for young children and if so, what form it should take. One view, widely adopted by educators of young children, is that direct instruction is inappropriate, and that programming for young children should be purely play-based – that ideas and learning should emerge from the children's play with limited teacher intervention.

Current research disagrees with this view. Balfanz (1999) "proposes that intentional teaching of mathematics to young children is both appropriate and desirable" (10). Ginsburg & Ertle (2008), citing Bowman, Donovan & Burns (2001), write that "it has become increasingly evident that free play is not sufficient to promote solid mathematics learning in many children, particularly the poor, who have the greatest need" (45). The value of play is not under question; it is certainly acknowledged by researchers in the field that "play provides valuable opportunities to explore and to undertake activities that can be surprisingly sophisticated from a mathematical point of view" (Ginsburg, 2006; Hirsch, 1996; in Ginsburg & Ertle, 2008, 45). However, play alone does not guarantee mathematical learning will take place: "play is not enough. It does not usually help children to mathematize – to interpret their experiences in explicitly mathematical form and understand the relations between the two" (Ginsburg, Lee & Boyd, 2008, 7). As Seo & Ginsburg (2004) acknowledge, "children do learn from play, but it appears that they can learn much more with artful guidance and challenging activities provided by their teachers" (103).

Even the purposeful selection of materials, intended to draw out mathematical ideas in children's play, do not ensure that the intended learning experiences will take place:

> …the presence of manipulatives alone in a free play context does not guarantee an educative experience …simply providing manipulatives without a purpose, direction, guidance, or feedback may not promote conceptual understanding, computational fluency, or strategic mathematical thinking. Manipulatives may be most useful when children have a purpose of their own or an adult creates one and when children reflect on their use or peers or adults cause them to do so. (Baroody, 2006, 204)

Purposeful teacher decisions and interactions are required to ensure meaningful mathematical experiences and learning for young children. deVries, Thomas & Warren (2010) consider whether play-based contexts and mathematics instruction need be mutually exclusive. These authors write: "play is a pedagogical tool that can enable learning and this learning can be maximized with appropriate, timely and effective adult input" (719). Weisberg et al. (2015) also point to studies that show that a combination of direct instruction and guided play (for example, showing children one strategy while reminding them that there may be other ways to solve the problem or explore) offer even more benefits for children. While their article examines primarily pre-school contexts, they also point to other research that demonstrates that older students who engage in a problem-solving activity before instruction learn more than students who engage in the standard progression of instruction followed by practice (Schwartz, Sears, & Bransford, 2005). These authors remind us that the balance between freedom and structure is critical to children's learning, and suggest that play can be seen as a metaphor for "any kind of activity that engenders active, engaged participation" (p. 11).

This is consistent with the findings from Fisher, Hirsh-Pasek, Newcombe and Golinkoff (2013) that point to guided play as the most effective mode of instruction for early mathematics learning. In this study of 70 four- and five-year-old children, researchers examined geometry learning outcomes in a unit in which children were introduced to properties of different shapes, including regular and irregular triangles, rectangles, pentagons and hexagons. The children were randomly assigned to one of three different groups. In one group, a free-play approach was used to introduce the concepts. The second group was a guided-play group, and in the third group, direct instruction was used to teach about the shapes. In the free-play group, the educator simply made the shapes available for children to use in their play. In the direct instruction group, the educator told the children about the properties of the shapes. In the guided-play group, children were asked to help discover the “secrets” of the different shapes. Interestingly, children in the free-play group did not show any changes in their understanding of shapes. Children in both the guided-play and direct-instruction groups both showed progress, but students in the guided-play group made significantly more progress and showed deeper understanding of the properties of shapes, compared to students who were taught through direct instruction. This has profound implications for the importance of guided play as an instructional approach.

In a study recently conducted in Ontario (Bruce & Flynn, 2012), researchers used a continuum of play and instruction developed by Baroody et al. (2006), and found that in the course of one lesson or activity, effective early mathematics educators may dip into different types of teaching. Baroody et al. (2006) developed a continuum of four types of teaching: traditional direct-instruction, guided discovery learning via an adult-initiated task, flexible guided discovery learning via a child-initiated task, and unguided discovery learning via a child-initiated task. Guided and flexible guided discovery learning appeared to be the most promising according to Baroody et al. Keeping this framework in mind, Bruce and Flynn examined more closely the two middle categories (guided discovery learning via an adult-initiated task, flexible guided discovery learning via a child-initiated task), and tested these out in classrooms to amplify the relationships between play, instruction, and ‘guided’ discovery.

Figure 1 reflects the researchers observations of the play-instruction continuum, which may occur at different times in the classroom or even at different points in the course of a single lesson. Based on their observations of teachers in the Ontario lesson study project, researchers concluded that Baroody’s continuum is not actually a sequence of locked strategies, but rather a set of strategies that can be drawn on, and flexibly arranged to maximize student learning in mathematics. Another way to consider these types of teaching would be to understand them as a matrix of overlapping strategies that respond to the teaching/learning moment and student needs.

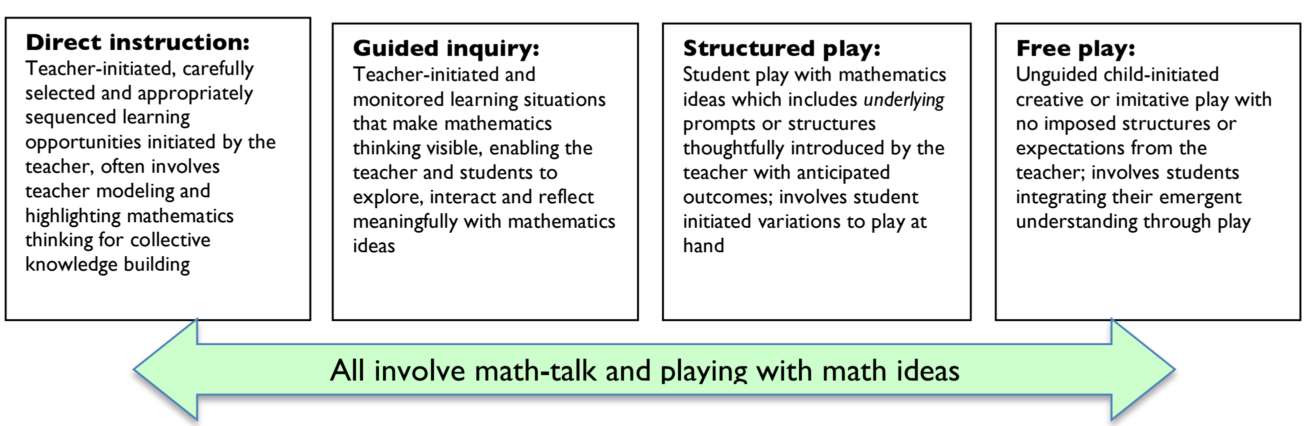

Figure 1. Range of effective teaching approaches for young children (Bruce & Flynn, 2012)

An illustrative example from this study relates to a team of teacher-researchers inquiring into the composition and decomposition of numbers. To begin, they observed students playing (unguided discovery) and gathered additional information through task-based interviews (guided discovery), which showed that children were able to count but had greater difficulty knowing more and less than five and how to make five. Subsequently, the team designed some fairly structured lessons (direct instruction with guided discovery) about composing and decomposing five using fingers as well as five frames using two colours. As a follow up in the form of structured play, a play station was designed ("The Five Bakery"), where "customers" used five frames and two colours to make orders and the "bakers" took the orders and created the cookies using play dough and glass beads for toppings. The bakery used structures from the lesson to support both the mathematical thinking and the independent play. With no adult intervention, students were then able to play at the bakery independently and with peers to further explore quantities of five (flexible guided discovery). Using this example, the researchers show how instruction is informed by student play, and is simultaneously being used to inform the design and structures of play-based activities, leading to an integration of instruction and play that questions the assumption that these approaches exist at opposite ends of a continuum of instructional approaches for young children.

The importance of exposure to mathematical language at home and at school

As the literature has demonstrated, the mathematical input children receive (i.e., the experiences to which they are exposed at home and school) has been identified as a factor in school success and is associated with the trajectory of growth of later mathematics abilities (Levine et al., 2010; Rudd et al., 2008; Klibanoff et al., 2006). There are two primary sources for this input leading up to and including the school years: parental and teacher math-talk. The importance of this input cannot be overstated, as the development of number concepts relies on a transition from nonverbal to verbal cognition. As stated by Ginsburg, Lee & Boyd (2008), "language is clearly deeply embedded in mathematics learning and teaching" (5). Further underscoring this deep connection is the evidence that more than half of the children who have difficulties in mathematics also have language and reading difficulties (and this percentage is even higher among low-income children) (Jordan & Levine, 2009).

In a longitudinal study, Levine et al. (2010) explored the effect of mathematical interactions between primary caregivers and their children in naturalistic settings. In particular, they examined how parental math-talk contributed to the acquisition of the cardinality principle. By the time the children were 30 months old, counting and labelling cardinal value sets became the focus of parent-child number talk. Although in the later months, parental math talk concentrated heavily on labelling sets of quantities, children's own dialogue focused predominantly on counting, showing that reciting count strings was acquired before the understanding of cardinality. Parents showed marked variability in the frequency of math-talk with their children. This variability was a significant predictor of their children's knowledge of cardinality. From a linguistic standpoint, this robust association between parental math-talk and knowledge of cardinality could be due to directing children's attention to labeled quantities and encouraging comparisons between them.

In one exceptional study about the affordances of mathematical language, Casey et al. (2008) used block-building interventions to examine effects on the development of spatial reasoning skills in kindergartners. There were two intervention conditions – mathematical play with rich language in the form of story, and mathematical play without story – and one control condition (no block play). Children who played within the story (i.e., language) context showed improved performance compared to the two other conditions (block play without story, and no block play). Researchers concluded that storytelling (i.e., language) provided an effective context for teaching spatial sense.

Klibanoff et al. (2006) sought to determine if teacher-directed math-talk in a typical classroom setting impacted the growth of children's mathematical knowledge throughout a school year. Past studies had demonstrated that children generally acquired words they hear more frequently (Hoff & Naigles, 1998), leading these researchers to hypothesize that children's acquisition of mathematics vocabulary would be influenced by the amount of math-talk to which they were exposed.

Children's mathematical knowledge was assessed both at the beginning and end of the pre-school year by Klibanoff et al. (2006). Teacher-directed number talk was audiotaped at two points in time throughout the year and coded for different mathematical concepts. Three main findings emerged from this study. Firstly, SES was a predicting factor for the level of children's mathematical knowledge throughout the school year. That is, those children from high- and middle-SES backgrounds displayed more knowledge compared to economically disadvantaged children at both points of assessment. Secondly, the amount of math-talk teachers provided also varied drastically from teacher to teacher. And thirdly, teacher-initiated number talk was significantly correlated with the growth of children's mathematical knowledge throughout the school year. The researchers noted that this last, and most important, finding is consistent with a general trend in linguistics research that shows that vocabulary growth is directly influenced by the amount of language input. Notably, *only mathematical language input rather than general language input was directly linked to children's growth in mathematical knowledge* underscoring the importance of math-talk in the early years classroom. It can be concluded that, along with primary caregivers, teachers can enhance the mathematical knowledge of children by initiating and fostering math-talk in the classroom (Klibanoff, et al., 2006).

Why might language development be so important to the development of children's mathematics? Children's early number sense, according to recent research summarized by Jordan & Levine (2009), can be categorized as primary preverbal number knowledge and secondary verbal or symbolic number knowledge. It is preverbal number knowledge that allows infants to subitize quantities (rapidly identify quantity without needing to count) less than four. Children need verbal access to the list of counting numbers to be able to deal with numbers larger than four with precision. It is verbal counting that allows children to eventually map numbers onto objects and move towards symbolic number knowledge (Jordan & Levine, 2009).

Overall, a robust link between caregiver/teacher mathematical talk and the growth of children's mathematical knowledge in the early years has been found (Levine et al., 2010; Rudd et al., 2008; Klibanoff et al., 2006). More specifically, mathematical talk initiated and encouraged by adults engaging with children fosters the understanding of important mathematical concepts, such as cardinality, and contributes to the growth of conventional mathematical knowledge. The quality of children's interactions with teachers is especially important given the variability in home environments.

Section 7: Specific mathematics programs that help young children

Early years math: Major programs

Three leading names in mathematics research – Herbert Ginsburg, Doug Clements and Sharon Griffin – have designed research-based mathematics programs for young children, respectively. Each of these programs: *Big Math for Little Kids*, *Building Blocks*, and *Number Worlds*, is reviewed below. In terms of curriculum content, *Big Math for Little Kids* is the broadest, covering six major curriculum strands, followed by *Building Blocks*, which emphasizes spatial and geometric concepts as well as number concepts, and *Number Worlds* is the narrowest, placing the primary emphasis on number.

Big Math for Little Kids

Big Math for Little Kids, as its name suggests, introduces young children to the "big ideas" of mathematics. Herbert Ginsburg, introduced the term *everyday mathematics*, which includes "informal ideas of more and less, taking away, shape, size, location, pattern and position" (Ginsburg, Lee & Boyd, 2008, 3). Even though children's thinking may be largely concrete at a young age, Ginsburg notes that they are also capable of abstract thought.

Big Math for Little Kids: Principles that guide program development

1. Build on children's knowledge and interests
2. Integrate mathematics into routine class activities
3. Introduce and enrich ideas in a planned way
4. Develop complex mathematical ideas
5. Promote language development and reflection
6. Encourage thinking like a mathematician
7. Provide repetition

During the development and field testing of the games and activities in the *Big Math for Little Kids* program with pre-kindergarten and kindergarten children, Greenes et al. (2004) made the following observations:

> "Children's abilities to anticipate future events, to predict outcomes and to think in conditional terms may develop with repeated and highly motivated participation in mathematical games…this type of "abstract" thinking is not usually thought to be characteristic of young children." (164)
>
> "In brief, our general hypothesis is that extensive engagement in activities offered by an intensely rich mathematical environment may lead to higher levels of competence than ordinarily observed in young children." (164)

Big Math for Little Kids: Curriculum strands

1. Number: labels and measurements as a way to quantify how many
2. Shape: recognition of two- and three-dimensional shapes
3. Measurement: comparison, standard measure and seriation
4. Operating on numbers: ways in which groups of objects can be put together and taken apart in preparation for the more formal exploration of operations
5. Patterns and logic: shape, number, color, pitch and rhythmic patterns
6. Navigation and spatial concepts: understanding of spatial vocabulary, such as *up, down, above, in front of, next to, between* and *to the right*

Storybooks were developed for each of the strands at both the pre-kindergarten and kindergarten levels.

Building Blocks

Building Blocks is an integrated program incorporating manipulatives, computers and print that is based on the learning trajectories developed by Doug Clements and Julie Sarama. The activities included in *Building Blocks* are aimed at “finding the mathematics in and developing mathematics from, children’s activity” (Clements & Sarama, 2007, 6). Based on their research, the program is organized into two areas: spatial and geometric competencies and concepts, and numeric and quantitative concepts. Emphasis was placed on “topics that were mathematically foundational, generative for, and interesting to young children” (6). In working with shapes, for example, children are asked to “act on” shapes (either on the computer or manually) according to their level. Activities are designed to support the child’s progression from one level to the subsequent one (i.e., from Pre-Composer to Piece Assembler).

Mathematical Competencies: Building Blocks

Two mathematical areas:

(a) spatial and geometric competencies and concepts; and
(b) numeric and quantitative concepts.

Three mathematical sub-themes:

(a) patterns and functions;
(b) data; and
(c) discrete mathematics (classifying, sorting and sequencing).

Number Worlds

Sharon Griffin, who contributed to the National Research Council’s report (2009), designed the *Number Worlds* program, in line with her research underlining the importance of numeracy to children’s mathematical understanding. The *Number Worlds* program (originally called *Rightstart*) is underpinned by the central conceptual structure theory. For a 6-year-old child developing an understanding of number, the central conceptual structure is an internal number line, a continuum increasing in 1-unit steps, to which the child has access when solving mathematical tasks and learning new ideas (Case & Okamoto, 1996). Central to this theory is that new ideas that are acquired must connect to the current knowledge of the child. The program is designed to recognize and adapt to the different developmental levels and needs of each child.

Number Worlds: Five instructional principles
Principle 1: Build upon children's current knowledge.
Principle 2: Follow the natural developmental progression when selecting new knowledge to be taught.
Principle 3: Teach computational fluency as well as conceptual understanding.
Principle 4: Provide plenty of opportunity for hands-on exploration, problem-solving and communication.
Principle 5: Expose children to the major ways number is represented and talked about in developed societies.
Other programs include:

Other important early math programs

Pre-K Mathematics Curriculum

The *Pre-K Mathematics Curriculum* (Klein & Starkey, 2002) includes 29 small-group preschool classroom activities employing manipulatives and 18 home activities for parents to use with their children. On average, teachers introduce one new activity each week during whole-group circle time and have children participate in each activity twice, in groups of 4-6, for approximately 20 minutes. "The activities are designed to be sensitive to the developmental needs of individual children. Suggestions are provided for scaffolding children who experience difficulty…" (Klein, Starkey, Clements, & Sarama, 2007, 5). The program also made use of the *DLM Express* software (Clements & Sarama, 2003), an earlier version of the *Building Blocks* software discussed above. Evaluation research showed impressive gains, with large effect size, for low-SES children who participate in the program.
Program content

1. Number and operations;
2. Space, geometry;
3. Pattern, measurement and data; and
4. Logical reasoning.

Storytelling Sagas

Storytelling Sagas (Casey, Kersh & Young, 2004) is a series of specially created supplementary mathematics storybooks for preschool through grade 2. Each of the six books focuses on a different content area (such as space, pattern or measurement) and combines oral storytelling with hands-on activity. The books all have a strong visualization/spatial reasoning component. The series of books obviously stresses the very important role of language as it involves children in active learning of mathematics. Evaluations of the program are underway. One study showed that embedding mathematics activities in stories is an effective pedagogical method for promoting spatial reasoning in a sample of low-SES kindergarten children (Casey, Erkut, Ceder & Young, 2008).

What these programs have in common, which may lead to success, is that they include: 1) regularity (of mathematics on a daily basis); 2) emphasis of teacher-student interaction (over peer interactions); and, 3) de-emphasis of student rotation through centres. In addition, these programs have in common the following characteristics:

- Utilization of multiple teaching contexts (whole class, small groups);
- Scaffolding of student learning and tailoring of activities to student needs with classroom- and home-based components; and
- Intensive professional development (a high degree of teacher support with *Building Blocks* being the highest with nine days of training and 16 hours of on-site coaching).

Math for Young Children (M4YC): A Classroom-based Spatial Reasoning Intervention

The Math for Young Children (M4YC) is a Canadian research program focused on working in classrooms with teachers and children (ages 4-7) to support mathematics learning through a spatialized approach to instruction. Currently in its fifth year, the M4YC project has involved working with over 200 teachers and 2000 K-2 students through an iterative design research approach to teacher professional development. The approach involves teachers and researchers working together throughout the school year to collaboratively design, implement, field-test and refine engaging and playful spatial reasoning lessons and activities. The research and professional learning program was developed in response to: 1) a growing recognition of the importance of spatial reasoning for mathematical learning and development (Verdine et al., 2014); 2) widespread neglect of spatial reasoning in early years mathematics (Clements & Sarama, 2011); 3) the finding that spatial thinking is a highly malleable construct (Uttal et al., 2013); and 4) the increasing recognition that spatial reasoning provides a means of offering children new opportunities and entry points into mathematics (Mulligan, 2015). Findings to-date have been encouraging. Researchers implemented the M4YC model in two separate Ontario school districts during one school year (a 5-month intervention at one site, a 9-month intervention at the other). Utilizing a quasi-experimental research design, 8 schools were either assigned to the experimental or control group. All participating children (N = 181) took part in identical pre- and post-tests, including measures of children's spatial reasoning, basic numerical skills, and mathematics achievement in geometry and numeration. Results revealed that in comparison to the control groups, children in the experimental classrooms demonstrated significant gains on tests of visual-spatial reasoning, 2D mental rotation, symbolic number comparison, and tests of mathematical achievement in geometry and numeration. These findings indicate the potential benefits of attending to young children's spatial thinking as a central component of early years mathematics instruction. Through integrating spatial training as part of regular mathematics instruction, and attending to the inherently spatial aspects of mathematics, researchers have been able to demonstrate not only widespread change in children's spatial reasoning but also evidence of improvements in both basic and advanced numerical skills. These results speak to the potential power of the classroom intervention model as a vehicle for establishing authentic teacher-researcher collaborations that seek to bridge research and practice. This large-scale, longitudinal project has led to the development of the teacher resource, Taking Shape: Activities to Develop Geometric and Spatial Thinking, Grades K-2 (Moss, Bruce, Caswell, Flynn & Hawes, 2016).

Section 8: Conclusions, recommendations and future directions for research

Effective mathematics education for young children holds great promise for improving later school success. Research shows that young children develop extensive every day mathematical understandings and are capable of learning more and deep mathematics than previously assumed. The most urgent need is to support teachers of young children with effective professional learning opportunities that help them harness this potential.

Key findings from the literature review

- Children have the potential and desire to learn mathematics, even at an abstract and symbolic level.
- Early childhood mathematics education is more complex than usually assumed.
- The kinds of mathematics programs that have been shown to be most effective through the highest caliber of research studies (gold-standard) include a spectrum of activities from play through to small group work, structured learning opportunities and direct instruction. These structures for learning are combined for daily concentration on mathematics learning that is purposeful and mathematics specific.
- Research demonstrates intentional teaching involves active, deliberate and planned instruction yielding positive results for all students.
- Despite noble efforts in public education, children from low SES environments generally lag behind their higher SES peers and require direct intervention and/or support to close the gap. Without high quality early instruction in mathematics, these students, who start behind, will continue to be disadvantaged and the gap will persist or even widen over the course of schooling.
- Emerging research indicates that when mathematics is supplementary or embedded, rather than a primary focus, the effects are less positive in promoting children's mathematics learning. Mathematics learning and teaching are crucial aspects of student success.
- Educators have an appetite for professional learning that helps them learn to support young mathematicians but require opportunities to engage in deep mathematical content learning and pedagogical content learning.
- Providing children with a solid foundation in mathematics learning not only contributes to addressing long-term systemic inequities but also supports student success in later mathematics, the sciences, reading and with non-routine problem-solving situations.
- Although there is a growing body of research concerning how young children develop and learn mathematics, these findings do not seem to be well known nor fully realized in mathematics teaching for young children. To ensure that *all* children develop the mathematical foundation they need for academic and overall success, teachers, curriculum developers, district school board personnel, researchers and policy makers need to transform

their approaches to mathematics education by supporting, developing and implementing research-based practices.

- Spatial reasoning is a key area for future research, and current findings suggest an urgent need for explicit attention to spatial reasoning in curriculum development, as well as support for both teacher and student learning in spatial reasoning.

Recommendations for good practice

To achieve high quality mathematics education for young children, the following recommendations are drawn from the literature review, as well as from key materials, including NCTM and NAEYC materials and the National Research Council's 2009 report, *Mathematics Learning in Early Childhood: Paths towards excellence and equity*.

Classroom

1. Enhance children's natural interest in mathematics.

Playing with mathematical ideas is a crucial feature of children's learning and mathematics understanding. Children demonstrate a natural interest in mathematics, and use mathematical ideas and knowledge, often in surprisingly sophisticated ways to make sense of their worlds. Educators need to work with these assets that children bring:

> By capitalizing on such moments and by carefully planning a variety of experiences with mathematical ideas in mind, teachers cultivate and extend children's mathematical sense and interest. Because young children's experiences fundamentally shape their attitude toward mathematics, an engaging and encouraging climate for children's early encounters with mathematics is important. It is vital for young children to develop confidence in their ability to understand and use mathematics—in other words, to see mathematics as within their reach. In addition, positive experiences with using mathematics to solve problems help children to develop dispositions such as curiosity, imagination, flexibility, inventiveness, and persistence that contribute to their future success in and out of school. (NAEYC and NCTM, 2002, 4).

2. Provide children with opportunities to engage in deep interaction with key mathematical ideas.

It is well known in the literature that time allotted for mathematics is far less than time allotted for literacy in the early classroom. Traditionally the mathematics in programs for young children has been embedded in daily classroom routines. Effective mathematics programs include intentionally designed mathematical learning experiences that provide opportunities for children to explore mathematical concepts deeply.

3. Provide ample time for children to actively inquire into mathematical concepts through a *range* of appropriate experiences and teaching strategies

Mathematics for young children must offer opportunities, structures and tools for children to connect their intuitive mathematical thinking to more formal mathematics. This takes many forms, including large and small group learning situations, brief teacher-student interactions based on student need, learning centres or stations created by the educator or co-created with students, problem play (play opportunities structured around a mathematical problem), games that support mathematics thinking and concept attainment, fantasy play and abstract reasoning opportunities, direct instruction, and storied contexts. In these contexts, the educator can observe and assess student learning to inform next steps. Careful noticing of student activity forms the basis of assessment to support instructional decision-making.

Educator Support

4. Provide extensive professional learning

Educators of young children require sustained professional learning in mathematics, and in particularly in spatial reasoning, an area which has received little attention before now. Professional development programs should be clearly grounded in research and include collaborative inquiry models. Governments and school authorities must fund development of and research on new programs, as well as implementation of current ones.

Research Activity

5. Continue to support research on learning potential

There is limited research on what children can do in particularly rich and stimulating environments. Educators must challenge children in order to understand the true extent of their capabilities. Ginsburg calls for an intensification of research into mathematics teaching "that focuses not so much on what children know but on what they could know under stimulating conditions" (Ginsberg, 2006, 162). Further, Ginsburg, Lee & Boyd (2006) argue that the field doesn't yet have a full understanding of what children are capable of in mathematics. They suggest that views of high quality mathematics education for young children should be supported by research in atypical teaching and learning situations without setting limits on what children can do. Research studies such as the Math for Young Children project, which explores a "no-ceiling curriculum" (see Bruce, Moss & Flynn, 2013), continue to provide crucial contributions in this area.

6. Expand and further develop research methods on teacher knowledge

Over the past decade, research has focused on constructs such as MKT (mathematical knowledge for teaching), TCK (teacher content knowledge), PCKT (pedagogical content knowledge for teaching), etc. These constructs have been very important because the field is learning that the specific mathematical content knowledge of teachers makes a difference to student learning (see Hill, Rowan & Ball, 2005, for example). While these have been very helpful for the field, these kinds of constructs do not necessarily align with teaching mathematics for young children. To successfully support the teaching of early mathematics, research is required on teachers' views and interpretations of learning, learners, and teaching practice, as well as their understanding of curriculum and its underlying mathematics concepts.

7. Expand and further develop our understanding of models of professional learning for educators of young children

Although there is a growing literature on effective professional learning programs for mathematics educators in general (see Davis and Simpt, 2006, for example), less research has been conducted on determining which professional learning programs are aligned with the particular needs of educators of young children. This research would look at the effectiveness of lesson study, collaborative action research as well as other alternative models of professional learning at the in-service and preservice levels.

Future directions for research: Educational Neuroscience and the case of spatial reasoning

The burgeoning field of Mind, Brain and Education (MBE, also known as educational neuroscience) is a fascinating development in the world of educational research. Its emergence formalized by the launch of the journal, *Mind, Brain and Education* in 2007, MBE aims to bring together the fields of "biology, cognitive science, development and education in order to create a sound grounding of education in research" (Fisher, 2009, p. 3).

MBE can shed additional insights into long-standing dichotomies in mathematics education; for example, MBE research weighs in on the rote instruction vs. inquiry-based learning debate. Consistent with research findings on guided play, recent findings from MBE research have demonstrated that inquiry, when combined with support and opportunities to practice, is an efficient approach to teaching and learning. This research provides a scientific basis for constructivism, the theory that individuals must construct their own understanding through experience (Vygotsky, 1978). Other studies have affirmed that "rote learning alone produces a narrow and brittle form of knowledge" that an individual may be able to repeat but does not understand and is unable to apply (Devlin, 2010). This may be linked to the idea that direct instruction alone may be insufficient for fully developing children's mathematical potential. MBE continues to blow the ceilings off of such ideas as we learn more about the amazing capabilities of the brain, along with the acknowledgement that there is so much more to know. Even more so, new understandings about brain plasticity affirm the importance of experience and education; they also provide strong evidence that intelligence and ability may be impacted by genetics and brain physiology, but are not necessarily pre-determined by them. Kurt Fisher (2009), in a seminal article that lays out the new and emerging field of MBE, writes:

> …cognitive and neuroscience research shows that knowledge is based in activity. When animals and people do things in their worlds, they shape their behaviour. Based on brain research, we know that likewise they literally shape the anatomy and physiology of their brains (and bodies). When we actively control our experience, that experience sculpts the way

> that our brains work, changing neurons, synapses, and brain activity (Hubel & Wiesel, 1970; Singer, 1995). (p. 5)

A concrete example of an area where MBE is impacting curricula and teaching lies in the case of spatial reasoning. Spatial reasoning has long been of interest to cognitive psychologists, but as noted above, has been largely ignored in the world of education. This is changing. Researchers across disciplines are beginning to come together from psychology, education and other disciplines to share knowledge and conduct research to better understand spatial reasoning and the implications for the classroom (Davis, 2015). The Ontario Ministry of Education is providing information and support for teachers to learn more about spatial reasoning (see Flynn & Hawes, 2014). Some of the impetus for this comes from excitement about recent discoveries that show that spatial reasoning is not a fixed trait as once believed (IQ test items are based on this belief), but that it is malleable and can be improved at any age (Uttal et al., 2013). In the case of spatial reasoning, we see a concrete example of MBE research – and the collaboration between cognitive psychologists, brain researchers and educators – in action. In September, 2016, the International Mind, Brain and Education Society will hold its conference in Toronto, and researchers along with teachers involved in the Math for Young Children project will share findings along with their experiences of collaboration and insights into education learned through the process.

The collaborative nature of science and education in the MBE movement cannot be overstated; MBE is an interdisciplinary approach, where the two-way exchange of new information is valued. Fisher (2009) provides an illustration of this two-way information flow between brain science and education (learning sciences) using the example of research on children who have had hemispherectomies (half of the brain removed for medical reasons, e.g., to stop severe seizures occurring in one half of the brain). In these cases, biological information about the processing capabilities of the left vs. right hemisphere helps to inform rehabilitation and support services, including education supports and strategies. But surprisingly, this is not the end of the story. In two case studies described by Fisher, children who had strong support and resources in place showed "remarkable plasticity in their learning and brain development. Despite having lost an entire hemisphere, they learned what they were not supposed to be capable of" (p. 7). According to the neurological understanding at the time, these children were not supposed to ever be able to learn to speak, or to draw, for example, yet these children surprised neuroscientists as they learned how to do both with support. In this way, education informed important new understandings about brain plasticity – the ability of the brain to form new and alternative pathways to perform tasks.

What is the role of the teacher in a framework that is informed by educational neuroscience? In the collaborative research paradigm of MBE, the teacher can be seen as an "educational translator or engineer [who] can help apply findings from cognitive science and neuroscience to learning in classrooms and can engineer educational materials and activities grounded in research that promote learning" (Fisher, 2009, p. 13). This is an empowering image of the teacher. In this role, the teacher is seen as an expert on children's thinking and development, informed and empowered by research to support student learning. Perhaps most importantly, MBE researchers see a powerful role for teachers in constructing researched knowledge, and see in MBE a model for bridging the research-practice gap in education. Fisher (2009) considers a long list of examples where research and practice come together to inform best practices, including medicine, meteorology, manufacturing, agriculture, construction, as well as the food processing, chemical and automotive industries. He notes that "somehow education has been mostly exempt from this grounding in research…If Revlon and Toyota can spend millions on research to create better products, how can schools continue to use alleged "best practices" without collecting evidence of what really works" (p. 4)? Importantly, the Mind, Brain and Education movement calls for deep collaboration between neurologists, cognitive scientists and educators, as well as students, so that educational practice and brain science can be mutually informed by research *in situ,* to best understand the challenges and opportunities in classrooms for student learning. Cognitive neuroscience is still a relatively young discipline (Devlin, 2010), but in the case of spatial reasoning, research in Ontario and elsewhere, is already proving to bring new insights into educational practice.

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