Mathematical Modelling and Its STEM Applications

In conversation:
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Summary: In this facilitation guide we provide ideas shared during the Professional Learning Network webinar presented by Drs. Marina Milner-Bolotin and Dragana Martinovic on August 15, 2022. The guide introduces contemporary research on modelling and its pedagogical potential in K-12 education and ideas for teaching modelling at various levels of schooling. We conclude with the overview and a working example of the educational framework for modelling in STEM subjects. The goal is to equip teachers with ideas for incorporating modelling into their mathematics and science curricula.

Introduction — Changes in mathematics curriculum

Questions: What are the noticeable trends in mathematics curriculum changes? How is teaching mathematics nowadays different, compared to teaching it according to the earlier curricula?

Those of you who taught other subjects probably remember situations in which your students had to use mathematics. This might have happened while you were discussing the abstract art of Kazimir Malevich, Theo van Doesburg, Pablo Picasso, or Piet Mondrian. You might have asked your students to name the figures on the painting, identify patterns, or compare the areas covered by different colours.

One secondary school teacher used "three abstract artists to show examples of the three main elements in abstract art: Piet Mondrian for line, Wassily Kandinsky for shape, and Mark Rothko for color. The only guidelines [she gave her] students were it must emphasize either line, shape, or color (all three will be included, but one will be showcased) as well as balance, unity, and a focal point. The three elements are the building blocks and the three principles are what make it a successful work of art." (Panetta, 2018).

Figure 1. Examples of works by Mondrian, Kandinsky and Rothko.

Did Panetta teach mathematics? No, but if Panetta's students had difficulties with any of the mathematical concepts used, she would have to clarify it to them.

This is an example of using mathematics in an *extra-mathematical world*[1], a useful term brought forward by Niss, Blum and Galbraith (2007), the term that is a more encompassing than the usual, *real world* (Fig. 1). One noticeable change in the current Ontario curriculum (Ministry of Education Ontario, 2020) is that knowledge that may belong to other subjects or is extracurricular, started finding place in mathematics classes—a reversal of what was previously happening in other subjects.

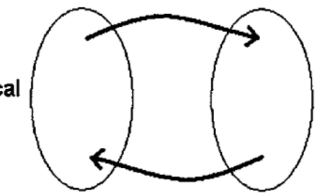

Figure 2. Visualisation of a modelling cycle (Niss, Blum, & Galbraith, 2007, p. 4).

Questions: Are you sometimes insecure about what to do in your STEM or other lessons, when your students have problems with concepts and skills they should have mastered in their mathematics classes? What do you do then? Teach it or skip it, since this is not "math time," or…?

Science teachers surveyed by Milner-Bolotin and Zazkis (2021) said that they are confused with what they can/may do with a mathematics content. Most of them avoided to use mathematics they perceived problematic for their students and attempted to "dumb it down." In terms of assessment, many science teachers felt that it would be unfair to assess language use or mathematics skills in other subjects.

> *In "a reversal of fortune" situation, teachers of mathematics are now asking how to treat non-mathematical knowledge when they implement, for example, coding or modelling in their classes. Teacher educators and facilitators of professional learning are asking what knowledge and skills (future) teachers need under these different circumstances.*

Question: How is working with mathematical models different from modelling?

One distinction is obvious; modelling is a process and models are things. But wait, isn't a number line a mathematical model? Or, a Cartesian coordinate system? What about manipulatives, such as the three-dimensional geometric solids (e.g., cube, pyramid, cone)? What about a quadratic formula or a graph of parabola? Are these models too? Yes, they are all models that are used under different circumstances and for various reasons. A model can be represented visually (using graphs, diagrams, sketches), verbally or symbolically[2]. Models could also be the (interim) results of mathematical modelling-the idea that we will explore further in this document (Martinovic & Milner-Bolotin, 2021).

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[1] "The extra-mathematical world can be another subject or discipline, an area of practice, a sphere of private or social life, etc. The term "real world" is often used to describe the world outside mathematics, even though, say, quantum physics or orbitals in chemistry may appear less than real to some. The extra-mathematical world [indicates] part of the wider 'real world' that is relevant to a particular issue or problem." (pp. 3-4).

[2] David Orlin Hestenes, a physicist and a science educator, writes that in both mathematics and science education, "A *model* is a representation of structure in a given system. A *system* is a set of related *objects*, which may be real or imaginary, physical or mental, simple or composite. The *structure* of a system is a set of relations among its objects. The system itself is called the *referent* of the model." (p. 17, 2010, emphases in the original).

Research on modelling in STEM context

After interviewing four Chinese mathematicians, five mathematics educators and seven mathematics teachers, Xu, Lu, and Yang (2022) found that "the mathematicians and mathematics educators tended towards the atomistic approach to the learning and teaching of modelling [i.e., focusing on analysing models mathematically], while the mathematics teachers placed greater emphasis on the holistic approach [i.e., implementing a full modelling cycle]. The mathematicians and mathematics educators stressed student autonomy [i.e., student-centred learning], while the teachers emphasised teacher demonstration [i.e., teacher-centred learning, especially among the senior teachers]." (p. 689).

Questions: When organizing a modelling activity in your class, do you rather demonstrate or let the student do most work? Can you share examples of a holistic vs. atomistic approach to modelling?

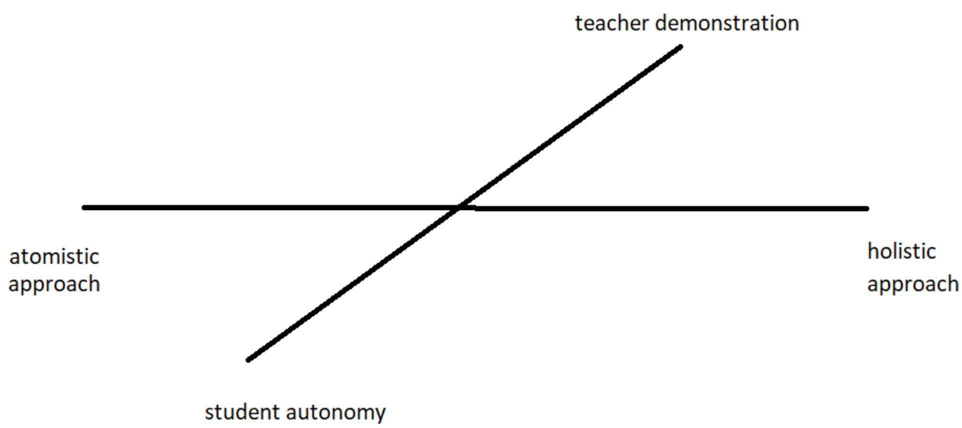

Figure 3. Distinctions in approaches to modelling (Xu et al., 2022)

These distinctions may be a consequence of the traditional teacher-centred pedagogy in China, but nevertheless present ways in which educators perceive modelling. The framework developed by Martinovic and Milner-Bolotin (2021; Appendix A) assumes that teachers release control during modelling activities, in accordance with a student-centred pedagogy in most modern curricula. Also, Xu et al. (2022) emphasise that neither the atomistic nor holistic approach is superior--both have a place in the classroom, as we will show using an example from the Ontario mathematics curriculum (Fig. 4).

Benefits of modelling

For mathematicians, scientists and engineers, modelling is a methodology, along with design and experimentation (Ortiz-Revilla et al., 2020). It may be seen as "the most relevant characteristic of the scientific mode of knowledge production" (p. 870).

Educational researchers are full of praises for modelling, as it leads to remarkable learning gains, especially in underserved student populations and students at risk, or as Lesh, Young, and Fennewald (2010) write, "modeling is virtually unparalleled in the successes that it has produced" (p. 283). Lesh and Yoon (2007) highlight how "models & modelling perspectives reject the notion that only a few exceptionally brilliant students are capable of developing significant mathematical concepts unless step-by-step guidance is provided by a teacher" (p. 163). These are good reasons to have modelling in the mathematics curriculum.

How is modelling different from problem solving? How is modelling in science different from modelling in mathematics?

According to Lesh and Yoon (2007), problem solving is creating a path from givens to the end-result, such that never leaves a world of mathematics (Fig. 1). Comparatively, "Model-eliciting activities […] are problem solving activities that elicit a model" (Lesh & Yoon, 2007, p. 163). Such activities need to demonstrate usefulness of the model in some real-life realm outside of mathematics.

The recent 2020 Ontario Grades 1-8 mathematics curriculum describes the process of mathematical modelling (Fig. 4). Teachers are asked to use modelling activities in all grades, as part of Algebra strand. In Fig. 4, components #1, #2, and #4 belong to the "real-life situation" circle (the extra-mathematical world referred by Niss et al., 2007), while #3 is distinct and belongs to the world of mathematics.

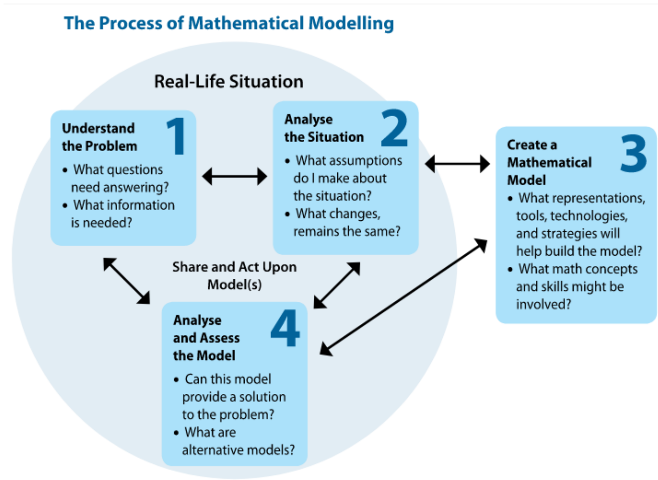

Figure 4. The process of mathematical modelling (Ministry of Education Ontario, 2020).

The numeration of components in Fig. 4 suggests that the modelling activity should be organized in a specific order and that it should preferably contain all the components. However, by noticing the three cycles, teachers could realize the similarities and differences to some already familiar pedagogies:

(a) *1-2-3-4 cycle:* A full modelling process takes students from a real-life situation (1, 2) to a mathematical model (3) and back to the real-life situation (4). The process is cyclical and iterative.

(b) *2-3-4 cycle:* This process is a simplified version and more akin to traditional problem-solving. The student seemingly has the content knowledge required to complete the task. However, if the analysis (4) shows that students need to re-assess the situation or re-do the model, they could be reverted to (2) or to (3).

(c) *1-2-4 cycle:* This process is entirely conducted in a real-life environment and may be organized in situations when students are given models to choose from and assess (Martinovic & Milner-Bolotin, 2021). For example, the teacher can ask what kind of regression best describes the data.

The bidirectional arrows in the diagram emphasize that despite the numbering, skipping a step or reverting to a previous one is possible. This opens opportunities for teachers to design flexible activities that address different learning goals. At the same time, the diagram focusses on what the students should be doing during the modelling activity, but it completely ignores the teacher. As a result, teachers may not see it as relevant or misinterpret it (Barquero, Bosch, & Romo, 2018; Frejd, 2012). For these reasons we will next discuss different roles during modelling.

Teacher's and students' roles during modelling

Mathematical modelling is challenging for both students and teachers (Xu et al., 2022). The main issue arises from its reliance on competences that deviate from those that belong to the traditional mathematics curriculum. It is of no surprise then, that in classrooms there were often recorded evident departures from the curriculum expectations regarding the modelling process. In addition to challenges regarding the time needed to organize a full modelling cycle and assess it, the teachers are found to experience mathematical, pedagogical, and epistemological challenges (Manouchehri, 2017). Based on Manouchehri's analyses of a 25-hours long professional development sequence (PD) with the middle and high school teachers in the US, mathematical challenges included difficulties of identifying variables, what information to keep, when to approximate and use heuristics, and when to use formulas and exact algorithms. Pedagogical challenges included teachers' confusion about short- and long-term outcomes and assessment of modelling, especially in view of most modelling activities organized as teams work. And finally, epistemological difficulties relate to perceived subjectivity of the assessment of what is important and what is not: is the model adequate or needs more refinement? is the background knowledge adequate? and so on.

Lesh and Yoon (2007) highlight how "the modelling cycles that problem solvers go through generally involve systematically rethinking the nature of givens, goals, and relevant solutions steps - or patterns & relationships that are attributed to surface-level data. Therefore, the most significant things that are being analyzed and transformed (or processed) are students' own ways of thinking about givens and goals - and patterns and regularities that are attributed to (rather than being deduced from) the information that is available." (p. 167). Similar ideas are highlighted by Carrejo and Marshall (2007), who caution that if not adequately prepared, "many teachers may […may rely on] direct instruction methods that do not facilitate conceptual understanding or abstraction [, and even …] abandon an inquiry-based approach altogether." (p. 48). Pollak (2011) also finds it problematic when teachers are too prescriptive:

> The heart of mathematical modeling, as we have seen, is problem *finding* before problem *solving*. So often in mathematics, we say "prove the following theorem" or "solve the following problem." When we start at this point, we are ignoring the fact that finding the theorem or the right problem was a large part of the battle. By emphasizing the problem finding aspect, mathematical modeling brings back to mathematics education that aspect of our subject and greatly reinforces the unity of the total mathematical experience. (p. 64)

Modelling motion (kinematics)

Why physics? Why motion? How is kinematics related to mathematics? To answer these questions, we turn to the history of science. Brousseau (1997) writes, "Obstacles of really epistemological origin are those from which one neither can nor should escape, because of their formative role in the knowledge being sought. They can be found in the history of the concepts themselves" (p. 87). The idea is to draw on historical "arguments in order to choose a genesis of a concept suitable for use in schools and to construct or 'invent' teaching situations that will provide this genesis" (p. 96). Or, as Jankvist (2009) advises, "To really learn and master mathematics, one's mind must go through the same stages that mathematics has gone through during its evolution" (p. 239). Of course, that is not always possible, but whenever we face a particularly difficult concept, looking into its historical development may be a helpful approach.

Koetsier (2012) writes that, “Motion has always played a role in geometry” (p. 497). He describes how Euclid defined 3-dimensional figures as results of revolution of 2-dimensional figures (i.e., a sphere: revolution of a semi-circle about its diameter; a cone: revolution of a right-angled triangle about one of its legs; a cylinder: revolution of a rectangle about one of its sides). Similarly, Newton conceptualised mathematical quantities as the result of motion (i.e., lines created by a moving point, planes by the moving lines, solids as moving plane segments, angles as rotating sides). In that way, historically, uninterrupted (continuous) mathematical objects were results of embodied continuous motion (Radford, 2009). Probably drawing from these historical references, Hestenes (2010) sees the main deficiency of mathematics curriculum in that it lacks connection to physical intuition, which would supply “the structural links to bodily experience from which all meaning ultimately derives. He suggests organizing mathematics curriculum “*around models*, not topics!” and urges for students to,

> become familiar with *a small set of basic model[sic] as the content core* for each branch of science, along with selected extensions to more *complex models*… Understanding the structure starts with identifying elements in a system and their relations, and visually representing those. (p. 33)

Then the analysis starts; is the model linear or not? Validation could be done through an experiment and data collection. At the end of this phase, students should have “clear answers to two questions: What is their model, and how well does it work?” (p. 35).

Question: What are some basic models at the content core for a branch of science you are familiar with?

For educators, this makes perfect sense – we know that students learn best when they experience or embody taught concepts. When modelling kinematics phenomena, students could roll a ball on a flat surface and record their observations; roll it down and up a ramp; follow a movement of a ball tossed straight in the air or let go from a window, or a trajectory of a ball tossed under an angle. Similar activities are part of play regardless of one’s age, but this time students are asked to approach “play” methodically.

A French philosopher, Nicole Oresme (1323-1382), represented the variations of speed and time by means of geometrical figures, in which line AB represents **time** and the perpendicular lines constant (Fig. 5(a)) or uniformly increasing **speed** values (Fig. 5(b)). A uniform speed motion is represented as a rectangle – the consequence of it having equidistant (horizontal) bases (so AC=EF=BD), while a uniform non-zero acceleration is represented as a right-angle triangle, with similar triangles having proportional sides.

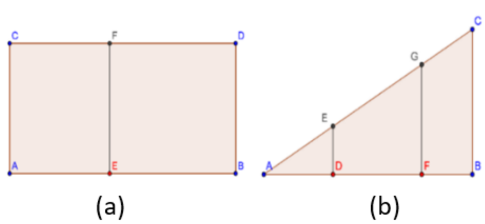

Figure 5. Oresme’s representation of uniform speed (a = 0) and uniform acceleration (a = const., $a \neq 0$) (Martínez et al., 2017, p. 1750).

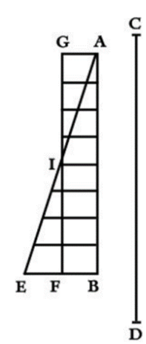

Figure 6. Uniformly accelerated motion – free fall (Galilei, 1954, Fig. 47.)

Galileo Galilei’s (1564-1642) representation of a uniformly accelerated motion during free fall (Fig. 6) combines a rectangle with a right-angle triangle. A body starts from rest at C and uniformly accelerates due to gravity. The timeline between two arbitrary points A and B is split into equal time intervals during which the speed uniformly increases from 0 to the length of EB. Given that F is the middle of EB, the rectangle AGFB has the area equal to the area of the triangle AEB, from which Galileo[3] concluded that, "The time in which any space is traversed by a body starting from rest and uniformly accelerated is equal to the time in which that same space would be traversed by the same body moving at a uniform speed whose value is the mean of the highest speed and the speed just before acceleration began" (p. 172).

The following motion formulas are part of a general physics curriculum. In them, r represents object's position, v and $\bar{v}$ represent its instantaneous and average velocity respectively, t stands for time, and r_0 and v_0 are initial values of position and velocity respectively.

1. Uniform motion ($v = const; a = 0$):

   a) $r(t) = r_0 + v_0 t$ and (b) $r(t) = r_0 + \bar{v}t$,

2. Motion with constant, non-zero acceleration (:

   c) $r(t) = r_0 + v_0 t + \frac{1}{2}at^2$ and (d) $r(t) = r_0 + \bar{v}$t

Possible Questions (Note: In the previous examples, the children were travelling in the positive direction, therefore their velocities were positive. These questions can be extended to discuss the meaning of positive and negative velocity and acceleration as well as the connection between velocity and speed):

1. *What is changing in these formulas and what remains the same? What does it mean?*
2. *What does the word "uniform" mean for each one of these cases?*
3. *How are the formulas (a) and (b) different? When would you use each one of these formulas?*
4. *Why do formulas (a) and (b) use the initial and average velocity respectively?*
5. *When is average velocity the same as the initial velocity and when are they different?*
6. *How could all these formulas be represented graphically?*
7. *Where does the ½ in the third formula come from? What is its significance?*
8. *Can you use formulas (c) and (d) to describe motion with constant velocity? Justify your answer.*
9. *How can you represent these formulas using a graphing software (e.g., Desmos graphing calculator)? What does it tell you about these relationships?*

Modelling example:

To describe motion, means to know object's position at all times. To do so, we must account for how fast and where it is moving and where it started from. For example, a car can move with the speed of 10 m/s northward or southward. While the car's speed is the same in both cases, its direction isn't. To describe this motion, we need to account for the direction, as well as the speed. Why is that so? If you drive for 10 minutes northward and then for 10 minutes southward (at the same speed), you will end at the same place you started; at that moment your car's *displacement* (change in position) will be 0, although you spent 20 minutes driving and travelled the *distance* of 12 km. Here is a worked-out example of a modelling activity suitable for students from mid-school and up. By the end of this lesson, the students will be able to distinguish between average vs. instantaneous speed, and reading a map vs. reading a graph, and comparing ratios.

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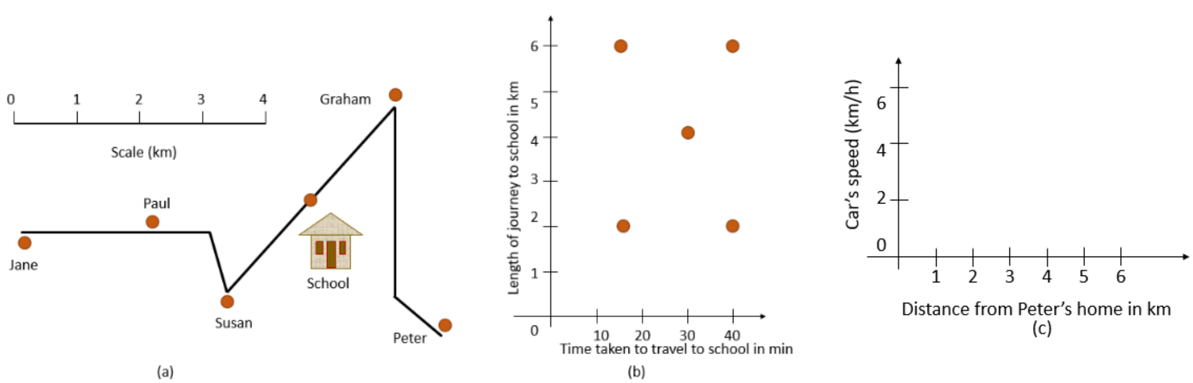
[3] More ideas on how to geometrically represent different ideas from kinematics, you can find in Galileo Galilei, Dialogues Concerning Two New Sciences, translated by Henry Crew & Alfonso de Salvio, with an introduction by Antonio Favaro, Dover Publications, Inc., New York, 1954: 153–243. Originally published in 1904 by the MacMillan.

Simplified situations could be shared with younger students, for example a similar ranking question was posed to Grade 3 students in Carmona and Greenstein's (2010) study. In the Team Ranking Problem, the students compared the performance of 12 teams whose wins and losses in the games were presented on the unitless Cartesian coordinate system. In other words, the students were presented with the coordinate system on which 12 points were scattered. Although the Cartesian coordinate system was not yet in the curriculum, the students developed a sophisticated ranking system between the teams, which included a kind of pre-slope understanding of a ratio between wins and losses, and in the case of a tie, using the total number of games played as the deciding factor.

Question 6. poses interesting possibilities for extending this modelling activity into Grade 9, where we also suggest adding the more conventional speed/time graphs, where students may realize how the same situation could be presented differently based on the labels on axes. For higher grades, this motion activity could be extended with ideas around vectors, displacement, and velocity. Also, technology (e.g., motion detectors, Desmos graphing calculator, and video cameras) could be used throughout the activity.

Example 1: Modelling Students' Motion *(adapted from Shell Centre for Mathematical Education, 1985, pp. 28-36)*

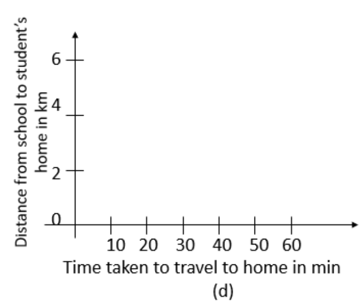
Jane, Graham, Susan, Paul, and Peter all travel to school along the same country road every morning. Peter goes in his dad's car, Jane cycles, and Susan walks. The other two students change their mode of travel daily.



Example 1: (a) The map of the country road. (b) The points represent students' **Monday travel**. (c) Coordinate system to describe Peter's dad's car motion on Monday.

Justify your answers and clearly state your assumptions for the questions below.

1. *Label each point on graph (b) with the name of the person it may represent.*
2. *How might have Paul and Graham traveled to school* ***on Monday****?*
3. *Peter's father can drive at 30 km/h on the straight sections of the road, but he has to slow down for the corners. Sketch a graph on coordinate system (c) to show how the car's speed might vary along the route to school.*
4. *According to your model, what was the car's* ***instantaneous*** *speed at 3.5 km from Peter's home?*
5. *Sketch a graph of the car's* ***average*** *speed along the road on the same coordinate system (c).*
6. *The average speed is calculated as a ratio of the distance traveled over the elapsed time. Based on the data given in (b), rank the students' average speed. Clearly state your assumptions.*
7. *All the students left school at 3 pm on Monday. Use coordinate system (d) to represent how students' distance from school varied with time, provided they used the same mode of transportation going home as coming to school.*

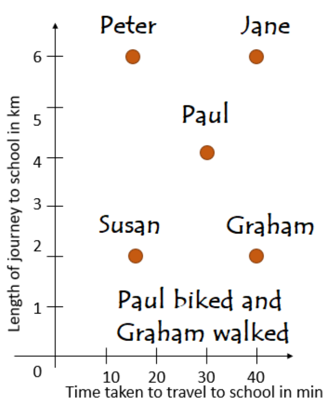


Stages of the activity (using Educational Framework for Modelling in STEM from Appendix A)

Stage I: <u>Teacher</u> has prepared the activity. It is a motion example that uses some new terms (e.g., instantaneous, and average speed). After anticipating difficulties and identifying learning goals, the teacher talks about the experimental evidence in science vs. providing a proof in mathematics. For example, the students will be asked to justify their results rather than prove them.

Stage II: <u>Teacher</u> provides a geographic map of the school area and handouts; encourages and records the discussion on the board; organizes students in groups, according to the scenario they chose. <u>Students</u> discuss ways in which they travel to school; on the map, they trace their usual travel paths, and use the scale to calculate the distance they travel daily. Based on it and the time it takes the students to get to school, they calculate their average speeds. Then, they try to understand Jane and her peers' travel arrangements (see Example 1(a)) and record their assumptions (e.g., the school area is flat or hilly; the road is made of asphalt or macadam) and the consequences on the handout.

Stage III: This stage focuses on students' contributions. The teacher monitors student discussion and assesses if they are ready to move on with creating their models (if they know how to read a map—important for understanding the *image (a)*). For example, they could answer *Questions 1.* and *2.* above in the following way:



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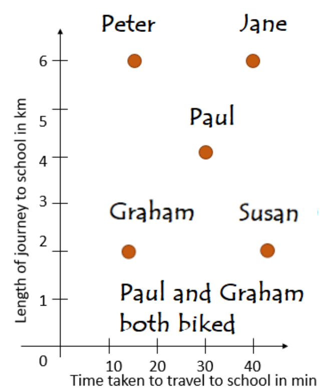
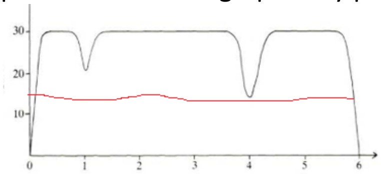


or something else!

Stage IV: Students work on initial modelling stages, the teacher facilitates.
Students may come up with the following models for *Question 3.*, which will determine their answer to *Question 4.* (red line represents the average speed anticipated by a student estimated according to the speed vs. distance graph they produced):



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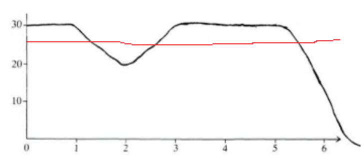

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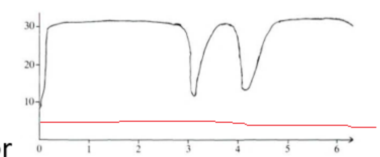

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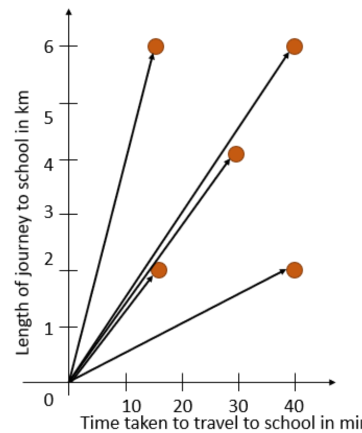
Stage V: Students test their models, the teacher facilitates.
Students compare their models; they find similarities and differences and relate them to assumptions or revise them if misunderstandings are noted. The teacher introduces definition of average speed and monitors students' approaches to completing *Question 6*. Students may come up with different criteria for ranking of which this one may emerge: Based on the values of ratios, students' average speeds are ranked according to the slopes of the lines (i.e., length of journey/time taken).

Stage VI: Knowledge consolidation facilitated by the teacher. To consolidate the knowledge, the teacher asks students to complete *Question 6*. The difference between instant and average speeds is reinforced. Different students' models may be revisited in detail with the help from the student groups that developed them.



(b)

Throughout the activity, the teacher pays attention to the difficulties known to be experienced by students (Doorman, 2005), including: -the difference in which the term "average" is used in physics compared to mathematics and -that for curves and lines (distance vs. time graph), the average rate of change (average speed) is calculated in the same way for both types of graphs, while for the instantaneous one (instantaneous speed), it is not (i.e., for lines in distance vs. time graphs, it is equal to one of the points coordinates in the corresponding speed vs. time graph, while for curves in the distance vs. time graph, it is the slope of a tangent to the same curve in that point).

Oresme's visualization (Figure 5 a) and b)) of relationship between uniform speed and time points to distance being equal to the area of a rectangle or a triangle in the case of uniformly accelerated motion, a concept difficult to understand since it equates the linear measure with the area. The fact that standing still is not represented as a point but rather as a horizontal line on a speed vs. time graph is also confusing. Motion detectors could help students to embody graphs as representations of their movement. Video recording motion and analyzing it collectively would help clarify or alleviate possible students' misunderstandings.

Additional resources (1-6, from Manouchehri, 2017)**:**

1. Rainfall problem (Bocci, F. (2012). *European Journal of Physics*, 33, 1321).
2. Basketball problem (Barrett, G., Markovich, K., & Compton, H. (1999). *Contemporary pre-calculus through applications (2nd ed., p. 275).* Glencoe/McGraw-Hill).
3. Spaghetti problem (D'Andrea, C., & Gomez, E. (2006). The broken spaghetti noodle. *The American Mathematical Monthly, 113(6)*, 555–557.
4. Financing college education (Dossey et al., (2003). *Mathematics methods and modelling for today's mathematics classroom (p. 97).* Pacific Grove, Thomson Learning).
5. Ping-pong ball problem (Starfield, A.M., Smith, K.A., & Bleloch, A.L. (1990). *How to model it: Problem solving for the computer age.* McGraw-Hill).
6. Establishing a new international airline hub.
7. Mathematics Assessment Project materials.
8. Arithmetic and Algebra to Solve Fairness Problems.
9. Modelling Instruction Program at the Arizona State University.

Appendix A: Educational Framework for Modelling in STEM (Martinovic and Milner-Bolotin, 2021).

Stage	Teacher's and students' roles during modelling*	Release of control
I	Teacher prepares students: Discusses how the epistemology is reflected in modelling. Teacher prepares a lesson: Selects a phenomenon, develops activities, anticipates difficulties, questions, challenges, etc.	Teacher releases control of student learning as students advance from Stage I to VI
II	Students get immersed in contextually rich concrete experiences; discuss how each member of the group can contribute. Teacher provides resources; records initial questions; organizes what students know and what they want to know; organizes groups; discusses what/how members of one's discipline come to know; guides/scaffolds modelling activities.	
III	Students contribute what they know and what knowledge the situation requires; assess how the present situation challenges and extends what they know. Teacher facilitates, provides resources, organizes students' initial questions, discusses limitations, monitors their work.	↓
IV	Students propose working hypotheses; propose patterns, models, theories, etc. that might explain the relationships between observed phenomena. They discuss and use different perspectives of the group members to interpret and understand the phenomenon. Teacher monitors students' work.	
V	Students test if and how the results of Stage IV agree with new concrete experiences; collect new evidence; use proposed models to generate new data, explain the phenomenon, allow for alternative explanations; test the models. Decide if the work is done and could be reported or start a new cycle of inquiry. Teacher monitors students' work, asks questions, and regroups students when required.	Students gain control of their learning as they advance from Stage I to VI
VI	Students return to Stage II with enhanced understanding of the phenomenon, with a set of new questions as a motivation for a new cycle. Teacher consolidates or revisits the modelling activity, suggests modifications for the follow-up.	

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